Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.14

14.1 Quantum nondemolition measurements: Back-action evasion of a meachanical oscillator using a cavity

Quantum nondemolition (QND) measurement is a special type of measurement of a quantum system in which the uncertainty of the measured observable does not increase from its measured value during the evolution of the system. This necessarily requires that the measurement process preserve the physical integrity of the measured system, and moreover places requirements on the relationship between the measured observable and the Hamiltonian of the system.

In order to start, first we define a QND observable. In general a QND variable ($\hat{A}$) is defined as an observable that commutes with itself at every time during the measurement:[1]

$$[\hat{A}(t_i), \hat{A}(t_j)] = 0 \tag{1}$$

If this condition is satisfied at all times (any t_i and t_j), then $\hat{A}$ is called a continuous QND observable; if it is satisfied only at special times, then $\hat{A}$ is a stroboscopic QND observable.

1. Consider a free particle with mass m. The Hamiltonian has the following form: $\hat{H} = \frac{\hat{p}^2}{2m}$. Show that momentum ($\hat{p}$) is a continuous QND observable in this system.

2. Now consider a harmonic oscillator and show that $\hat{x}$ and $\hat{p}$ are not continuous QND observables. Find the condition on measurement time so that these observables are stroboscopic QND.

3. For a harmonic oscillator one could define slowly varying amplitudes ($\hat{X}_1$ and $\hat{X}_2$ quadratures) as:

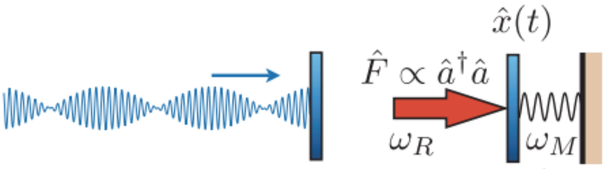
$$\hat{X}_1 = \hat{x}\cos(\omega_m t) - \frac{\hat{p}}{m\omega_m}\sin(\omega_m t) \tag{2}$$

$$\hat{X}_2 = \hat{x}\sin(\omega_m t) + \frac{\hat{p}}{m\omega_m}\cos(\omega_m t) \tag{3}$$

   where ω_m is the mechanical frequency and m is the mass of the oscillator.

   (a) Show that the quadratures are canonically conjugate variables such that $[\hat{X}_1, \hat{X}_2] = i\frac{\hbar}{m\omega_m}$

   (b) Here we use a detector-oscillator Hamiltonian: $\hat{H}_{\text{int}} = -\hbar g \hat{x}\hat{F}$, where g is the coupling strength and F in the cavity-position detector will be considered as the number of photons in the cavity[2].



[1] Braginsky, Vladimir B., Yuri I. Vorontsov, and Kip S. Thorne. "Quantum nondemolition measurements." Science 209.4456 (1980): 547-557.

[2] Similar to the radiation pressure coupling that was introduced in the previous homework

(c) Now consider the case where the coupling strength is modulated at the mechanical frequency ω_m such that $g(t) = g\cos(\omega_m t)$. Using this modulated coupling[3], recalculate the interaction Hamiltonian and average out the rotating terms at $2\omega_m$. Show that in this case your Hamiltonian commutes with $\hat{X}_1$ observable and so a QND measurement can be performed on this observable[4].

14.2 Dissipative squeezing

In this exercise we illustrate how we can generate squeezed states of a mechanical oscillator by driving an optomechanical cavity with two laser fields. We follow the method proposed in the following reference:

- *Arbitrarily large steady-state bosonic squeezing via dissipation*, Andreas Kronwald, Florian Marquardt, and Aashish A. Clerk *Phys. Rev. A 88, 063833*

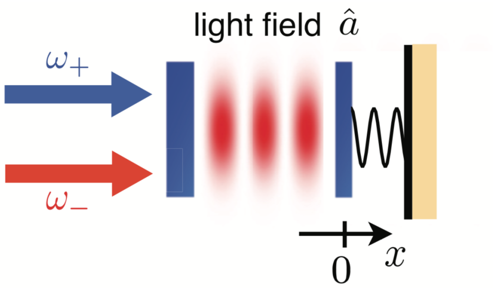


Figure 1: An optomechanical cavity is driven on the red and blue mechanical sideband with different laser amplitudes.

Consider a standard optomechanical system shown in Fig (1). It is described by the optomechanical Hamiltonian:

$$\hat{H}_{sys} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a}(\hat{b}^\dagger + \hat{b}) \tag{4}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity and the mechanical resonator respectively with corresponding resonance frequencies ω_c and Ω_m. g_0 is also the optomechanical coupling rate. As shown in Fig (1) the cavity is driven with two laser fields with different frequencies ω_+ and ω_-. The driving Hamiltonian of the system can be written as:

$$\begin{aligned}\hat{H}_{dr} &= \hbar\alpha(t)\hat{a}^\dagger + H.c. \\ &= \hbar(\alpha_+ e^{-i\omega_+ t} + \alpha_- e^{-i\omega_- t})\hat{a}^\dagger + H.c.\end{aligned}$$

The total Hamiltonian of the system is then written as $\hat{H} = \hat{H}_{sys} + \hat{H}_{dr}$.

1. Go to an an interaction picture with respect to the free cavity and mechanical oscillator Hamiltonian (i.e $\hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b}$).

2. Similar to the case of a single drive, the intracavity field will have an average coherent time dependence as well as fluctuations. Similarly for the mechanical displacement. Now we would like to find the *linearized* Hamiltonian describing the dynamics of the fluctuations. In our interaction picture, apply the transformations:

[3]This coupling can be achieved by simply modulating the amplitude of the light in the cavity

[4]Clerk, A. A., Florian Marquardt, and K. Jacobs. "Back-action evasion and squeezing of a mechanical resonator using a cavity detector." New Journal of Physics 10.9 (2008): 095010.

$$\hat{a} = \bar{a}(t) + \delta\hat{a} = \bar{a}_+ e^{i(\omega_c - \omega_+)t} + \bar{a}_- e^{i(\omega_c - \omega_-)t} + \delta\hat{a} \tag{5}$$

$$\hat{b} = \bar{b}(t) + \delta\hat{b} \tag{6}$$

For simplicity we assume that the two lasers are detuned exactly on the red and blue mechanical sidebands of the cavity, meaning: $\omega_\pm = \omega_c \pm \Omega_m$.

3. Show that the *linearized* Hamiltonian describing the dynamics of the fluctuations (i.e $\delta\hat{a}$ and $\delta\hat{b}$) has the form:

$$\hat{H}_{lin} = -\hbar\delta\hat{a}^\dagger(G_+\delta\hat{b}^\dagger + G_-\hat{b}) - \hbar\delta\hat{a}^\dagger(G_+\delta\hat{b}^\dagger e^{-2i\Omega_m t} + G_-\hat{b}e^{-2i\Omega_m t}) + H.c. \tag{7}$$

Find G_+ and G_-.

4. In the limit of a narrow linewidth cavity ($\kappa \ll \Omega_m$), we can neglect the time dependent terms in $\hat{H}_{lin}$ (also called as counter-rotating terms) since they get suppressed by the cavity. Now define a Bogoliubov-mode with annihilation operator

$$\hat{\beta} = \delta\hat{b}\cosh(r) + \delta\hat{b}\sinh(r) \tag{8}$$

Find r and $\mathcal{G}$ in such a way that $\hat{H}_{lin}$ can be rewritten as

$$\hat{H}_{lin} = -\hbar\mathcal{G}\delta\hat{a}^\dagger\hat{\beta} + H.c \tag{9}$$

5. Using a cooling process, known as optomechanical sideband cooling [5] [6], we can cool the mode $\hat{\beta}$ to its ground state. What is the ground state of the mode $\hat{\beta}$? (*Hint: you may refer to the exercise "Two-photon coherent states" in exercise series* 2)

[5] Quantum Theory of Cavity-Assisted Sideband Cooling of Mechanical Motion Florian Marquardt, Joe P. Chen, A. A. Clerk, and S. M. Girvin Phys. Rev. Lett. 99, 093902

[6] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Phys. Rev. Lett. 99, 093901 (2007)