Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.12

Solution: Wiener-Khinchin theorem

Proof:

$$\begin{aligned}
C_{XX}(\tau) &= \lim_{T\to} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau)\mathrm{d}t \\
&= \lim_{T\to} \frac{1}{T} \frac{1}{(2\pi)^2} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \mathrm{d}\omega X[\omega] e^{i\omega t} \int_{-\infty}^{\infty} \mathrm{d}\omega' X\left[\omega'\right] e^{i\omega'(t+\tau)} \mathrm{d}t \\
&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \mathrm{d}\omega' X[\omega] X\left[\omega'\right] e^{i\omega'\tau} \left(\lim_{T\to} \frac{1}{T} \int_{-T/2}^{T/2} e^{i(\omega+\omega')t}\, \mathrm{d}t \right) \\
&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \mathrm{d}\omega' X[\omega] X\left[\omega'\right] e^{i\omega'\tau} \delta\left(\omega+\omega'\right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega |X[\omega]|^2 e^{i\omega\tau}
\end{aligned}$$

Note that the correlation function and the power spectral density form a Fourier tranform pair.