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Generalized Phase-Space Distribution Functions*

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A set of quasi-probability distribution functions which give the correct quantum mechanical marginal distributions of position and momentum is studied. The phase-space distribution does not have to be bilinear in the state function. The Wigner distribution is a special case. A general relationship between the phase-space distribution functions and the rule of associating classical quantities to quantum mechanical operators is derived. This allows the writing of correspondence rules at will, of which the ones presently known are particular cases. The dynamics and other properties of the generalized phase-space distribution are considered.

1. INTRODUCTION

IN recent years the so-called phase-space formulation of quantum mechanics has found many applications, particularly in statistical mechanics[1] and in the study of the coherent properties of light.[2] Its basic feature is to permit one to calculate expectation values of quantum mechanical observables in the classical manner rather than through the operator formalism of quantum mechanics. That is, if

$$\langle \mathbf{G} \rangle = \int \psi^*(q)\mathbf{G}(\mathbf{q}, \mathbf{p})\psi(q)\, dq \tag{1.1}$$

is the expected value of the operator **G** than one attempts to write this as

$$\langle \mathbf{G} \rangle = \iint g(q, p)F(q, p)\, dq\, dp, \tag{1.2}$$

where $g(q, p)$ is the classical function from which the quantum mechanical **G** is obtained and $F(q, p)$ is the "distribution" function.[3]

The literature has dealt almost exclusively with the Wigner[4] distribution function. It is given by

$$F_W(q, p, t) = \frac{1}{2\pi} \int \psi^*(q - \tfrac{1}{2}\tau\hbar, t) \times e^{-i\tau p}\psi(q + \tfrac{1}{2}\tau\hbar, t)\, d\tau. \tag{1.3}$$

An integration over q or p yields the correct quantum mechanical marginal distributions. Unfortunately the Wigner distribution has some shortcomings. It can take on negative values and thus cannot be considered a true distribution. For this reason some authors have called it a quasi-probability distribution. More seriously, the Wigner distribution does not always yield the same expectation values as the correct quantum mechanical methods. For example, if one calculates the standard deviation of energy of the first excited state of the simple harmonic oscillator, using the classical Hamiltonian, a nonzero value is obtained. This is clearly in contradiction with the notion of an energy eigenstate.

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[1] For a general review of the Wigner distribution function the article "The Wigner Function and Transport Theory" by H. More, R. Oppenheim, and J. Ross which appears in *Studies in Statistical Mechanics*, V. De Boer and G. E. Uhlenbeck, Eds. (North-Holland Publishing Company, Amsterdam, 1962), Vol. 1, may be consulted. The article contains many references.

[2] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

[3] We restrict ourselves to a single dimension. Generalization to higher dimensions is direct. All integrals go from $-\infty$ to ∞.

[4] E. Wigner, Phys. Rev. **40**, 749 (1932).

Another distribution given by Margenau and Hill[5] in their work on joint probability distributions and later studied in detail by Mehta[6] is

$$F(q, p, t) = \frac{1}{4\pi} \text{Real } \psi(q, t) \times \int e^{-i\tau p}\psi^*(q - \tau\hbar)\, d\tau. \quad (1.4)$$

This distribution does give zero for the dispersion of energy for the ground and first excited states of the harmonic oscillator (although it gives nonzero values for the other eigenstates).

In the following we shall present a wide class of distribution functions of which the above two are special cases. The set of functions we propose to consider is

$$F(q, p, t; f) = \frac{1}{4\pi^2} \iiint e^{-i\theta q - i\tau p + i\theta u} f(\theta, \tau, t) \times \psi^*(u - \tfrac{1}{2}\tau\hbar, t)\psi(u + \tfrac{1}{2}\tau\hbar, t)\, d\theta\, d\tau\, du, \quad (1.5)$$

where $f(\theta, \tau, t)$ is any function[7] which satisfies

$$f(0, \tau, t) = f(\theta, 0, t) = 1. \quad (1.6)$$

It is readily seen that if $f = 1$, $\cos \frac{1}{2}\theta\tau\hbar$, we obtain (1.3) or (1.4) respectively. An integration of (1.5) over p or q yields the correct marginal distributions:

$$\int F(q, p, t; f)\, dp = |\psi(q, t)|^2, \quad (1.7)$$

$$\int F(q, p, t; f)\, dq = |\phi(p, t)|^2, \quad (1.8)$$

where $\phi(p, t)$ is the momentum state function. It should be noticed that f can be a functional of the state function itself. For example

$$f(\theta, \tau, t) = \int \psi(q - \theta\tau, t)\psi^*(q + \theta\tau, t)\, dq$$

satisfies (1.6). Thus F need not be bilinear in ψ.

2. CHARACTERISTIC FUNCTION AND CORRESPONDENCE RULES

The characteristic function $M(\theta, \tau, t)$ is defined as the Fourier transform of the distribution function.

$$M(\theta, \tau, t) = \iint F(q, p, t) e^{i\theta q + i\tau p}\, dq\, dp = \langle e^{i\theta q + i\tau p} \rangle, \quad (2.1)$$

which is the expected value of $e^{i\theta q + i\tau p}$. For a well-behaved distribution function, $M(\theta, \tau, t)$ contains as much information as the distribution function itself since (2.1) can be inverted to yield

$$F(q, p, t) = \frac{1}{4\pi^2} \iint M(\theta, \tau, t) e^{-i\theta q - i\tau p}\, d\theta\, d\tau = \frac{1}{4\pi^2} \iint \langle e^{i\theta q + i\tau p} \rangle e^{-i\theta q - i\tau p}\, d\theta\, d\tau. \quad (2.2)$$

By expanding $e^{i\theta q + i\tau p}$ of (2.1), $M(\theta, \tau, t)$ can be expressed in terms of the moments,

$$M(\theta, \tau, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(i\theta)^n (i\tau)^m}{n!\, m!} \langle q^n p^m \rangle_t. \quad (2.3)$$

Thus the problem of finding a quantum mechanical F reduces to finding the quantum mechanical equivalent of the classical quantity $e^{i\theta q + i\tau p}$ or $q^n p^m$. Rules which associate classical quantities to quantum mechanical operators are called correspondence rules or rules of association. There have been five such rules proposed. We list them here for completeness.

(a) Dirac's Rule of associating commutators with Poisson brackets.

(b) Von Neumann's Rules.

(c) Weyl's Rule.

$$e^{i\theta q + i\tau p} \rightarrow e^{i\theta \mathbf{q} + i\tau \mathbf{p}} \quad (2.4)$$

or equivalently as shown by McCoy[8]

$$q^n p^m \rightarrow \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} \mathbf{q}^{n-l} \mathbf{p}^m \mathbf{q}^l. \quad (2.5)$$

(d) Symmetrization Rule

$$e^{i\theta q + i\tau p} \rightarrow \tfrac{1}{2}\{e^{i\theta \mathbf{q}} e^{i\tau \mathbf{p}} + e^{i\tau \mathbf{p}} e^{i\theta \mathbf{q}}\} \quad (2.6)$$

or

$$q^n p^m \rightarrow \tfrac{1}{2}\{\mathbf{q}^n \mathbf{p}^m + \mathbf{p}^m \mathbf{q}^n\}. \quad (2.7)$$

(e) Rule of Born and Jordan[9]

$$q^n p^m \rightarrow \frac{1}{m+1} \sum_{l=0}^{m} \mathbf{p}^{m-l} \mathbf{q}^n \mathbf{p}^l. \quad (2.8)$$

The first four rules were considered by Schewell[10] and others. They have shown that (a) and (b) are self-contradictory. The Rule of Born and Jordan seems to have been forgotten. Moyal[11] has derived the Wigner distribution using the Weyl rule of association and Margenau and Hill have used the Symmetrization Rule to obtain (1.4). A straightforward calculation yields

$$F = \frac{2}{\hbar} \frac{1}{4\pi^2} \iiint \frac{e^{-i\tau p - i\theta q + i\theta u}}{\theta\tau} \sin \tfrac{1}{2}\theta\tau\hbar \times \psi^*(u - \tfrac{1}{2}\tau\hbar, t)\psi(u + \tfrac{1}{2}\tau\hbar, t)\, d\theta\, d\tau\, du, \quad (2.9)$$

[5] H. Margenau and R. N. Hill, Progr. Theoret. Phys. (Kyoto) 26, 722 (1961).
[6] C. L. Mehta, J. Math. Phys. 5, 677 (1964).
[7] f is assumed to be sufficiently well behaved so that the integrations in (1.5) can be interchanged. If f has the further property that $f^*(\theta, \tau) = f(-\theta, -\tau)$ than F will be real and the operator **G** of Sec. 2 will be Hermitian.
[8] N. H. McCoy, Proc. U. S. Natl. Acad. Sci. 18, 674 (1932).
[9] M. Born and P. Jordan, Z. Phys. 34, 873 (1925).
[10] J. R. Shewell, Am. J. Phys. 27, 16 (1959).
[11] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).

when the rule of Born and Jordan is used. This is a special case of (1.5) with $f = \sin(\frac{1}{2}\theta\tau\hbar)/\frac{1}{2}\theta\tau\hbar$. We shall now derive a fundamental relation which exists between correspondence rules and the general phase-space distribution (1.5). This will allow us to write, at will, correspondence rules of which the above are special cases.

Let $g(q, p)$ be a classical function and $\mathbf{G}(\mathbf{q}, \mathbf{p})$ the quantum mechanical operator corresponding to it. When (1.5) is used, the expected value of g is

$$\langle g(q, p)\rangle = \iint F(q, p, t)g(q, p)\,dq\,dp \tag{2.10}$$

$$= \iiint \gamma(\theta, \tau)f(\theta, \tau)e^{i\theta u}\psi^*(u - \tfrac{1}{2}\tau\hbar) \times \psi(u + \tfrac{1}{2}\tau\hbar)\,d\theta\,d\tau\,du, \tag{2.11}$$

where we have set

$$\gamma(\theta, \tau) = \frac{1}{4\pi^2}\iint g(q, p)e^{-i\theta q - i\tau p}\,dq\,dp. \tag{2.12}$$

Now consider the expression

$$\int \psi^*(q)e^{i\theta\mathbf{q} + i\tau\mathbf{p}}\psi(q)\,dq, \tag{2.13}$$

which equals

$$e^{\frac{1}{2}i\theta\tau\hbar}\int \psi^*(q)e^{i\theta q}\psi(q + \tau\hbar)\,dq, \tag{2.14}$$

since

$$e^{i\theta\mathbf{q} + i\tau\mathbf{p}} = e^{\frac{1}{2}i\theta\tau\hbar}e^{i\theta\mathbf{q}}e^{i\tau\mathbf{p}} \tag{2.15}$$

and

$$e^{i\tau\mathbf{p}}\psi(q) = \psi(q + \tau\hbar).$$

Making the substitution $u = q + \frac{1}{2}\tau\hbar$, we see that the expression (2.13) equals

$$\int \psi^*(u - \tfrac{1}{2}\tau\hbar)e^{i\theta u}\psi(u + \tfrac{1}{2}\tau\hbar)\,du. \tag{2.16}$$

Therefore (2.11) can be written in the form

$$\langle g(q, p)\rangle = \iiint \gamma(\theta, \tau)f(\theta, \tau) \times \psi^*(q)e^{i\theta\mathbf{q} + i\tau\mathbf{p}}\psi(q)\,d\theta\,d\tau\,dq. \tag{2.17}$$

We would like this to equal

$$\int \psi^*(q)\mathbf{G}(\mathbf{q}, \mathbf{p})\psi(q)\,dq = \langle \mathbf{G}(\mathbf{q}, \mathbf{p})\rangle. \tag{2.18}$$

Comparison then shows that the operator $\mathbf{G}$ can be taken as

$$\mathbf{G}(\mathbf{q}, \mathbf{p}) = \iint \gamma(\theta, \tau)f(\theta, \tau)e^{i\theta\mathbf{q} + i\tau\mathbf{p}}\,d\theta\,d\tau \tag{2.19}$$

$$= \iint e^{\frac{1}{2}i\theta\tau\hbar}\gamma(\theta, \tau)f(\theta, \tau)e^{i\theta\mathbf{q}}e^{i\tau\mathbf{p}}\,d\theta\,d\tau. \tag{2.20}$$

This establishes the method of obtaining the quantum mechanical $\mathbf{G}$ from the classical g if (2.18) and (2.10) are to yield the same value.

Equation (2.20) can be cast in a more operational form. Assume f to be analytic and expand it in a power series

$$f(\theta, \tau) = \sum_{rs} a_{rs}\theta^r\tau^s. \tag{2.21}$$

Equation (2.20) then becomes

$$\sum_{rsn}\frac{a_{rs}}{n!}(\tfrac{1}{2}i\hbar)^n\iint \theta^{n+r}\tau^{n+s}\gamma(\theta, \tau)e^{i\theta\mathbf{q}}e^{i\tau\mathbf{p}}\,d\theta\,d\tau. \tag{2.22}$$

But from 2.12

$$(\partial^{2n+r+s}/\partial q^{n+r}\partial p^{n+s})g(q, p) = i^{2n+r+s} \times \iint \theta^{n+r}\tau^{n+s}\gamma(\theta, \tau)e^{i\theta q + i\tau p}\,d\theta\,d\tau \tag{2.23}$$

or, upon inverting,

$$\theta^{n+r}\tau^{n+s}\gamma(\theta, \tau) = \frac{1}{i^{2n+r+s}}\frac{1}{4\pi^2} \times \iint\left\{\frac{\partial^{2n+r+s}}{\partial q^{n+r}\,\partial p^{n+s}}g(q, p)\right\}e^{-i\theta q - i\tau p}\,dq\,dp. \tag{2.24}$$

Substituting (2.24) in (2.22) we find

$$\mathbf{G}(\mathbf{q}, \mathbf{p}) = \frac{1}{4\pi^2}\sum_{rsn}\frac{a_{rs}(\frac{1}{2}i\hbar)^n}{n!\,i^{2n+r+s}} \times \iiiint\left\{\frac{\partial^{2n+r+s}}{\partial q'^{n+r}\,\partial p'^{n+s}}g(q', p')\right\} \times e^{-i\theta q' - i\tau p'}e^{i\theta\mathbf{q}}e^{i\tau\mathbf{p}}\,d\theta\,d\tau\,dq'dp'. \tag{2.25}$$

This expression for $\mathbf{G}$ is in normal form, that is, the $\mathbf{q}$ factors precede the $\mathbf{p}$ factors. We can therefore substitute ordinary variables q and p for the operators $\mathbf{q}$ and $\mathbf{p}$, perform the integration and resubstitute $\mathbf{q}$ and $\mathbf{p}$ for q and p, after the resulting integration has been put in a form so that the q factors precede the p factors. Carrying this out gives

$$f\left(\frac{1}{i}\frac{\partial}{\partial q}, \frac{1}{i}\frac{\partial}{\partial p}\right)e^{-\frac{1}{2}i\hbar(\partial^2/\partial q\partial p)}g(q, p). \tag{2.26}$$

Thus, to obtain a correspondence rule, chose any f which satisfies (1.6), and calculate (2.26). Then arrange the result so that the q factors precede the p factors and substitute $\mathbf{q}$ for q and $\mathbf{p}$ for p. The correspondence rules of Weyl, the rule of symmetrization, and the rule of Born and Jordan are obtained by taking f equal to 1, $\cos\frac{1}{2}\theta\tau\hbar$, $\sin(\frac{1}{2}\theta\tau\hbar)/\frac{1}{2}\theta\tau\hbar$, respectively.

3. PHASE-SPACE EIGENFUNCTIONS

If $\psi(q)$ is expanded in a complete orthonormal set $\{\varphi_n(q)\}$

$$\psi(q) = \sum_{n=0}^{\infty} a_n \varphi_n(q), \tag{3.1}$$

then the phase-space distribution function (1.5) takes the form

$$F(q, p, t; f) = \sum_{nm} a_n^*(t) a_m(t) h_{nm}(q, p; f), \tag{3.2}$$

where h_{nm} are the phase-space eigenfunctions and are given by

$$h_{nm}(q, p; f) = \frac{1}{4\pi^2} \iiint e^{-i\theta q - i\tau p + i\theta u} \times f(\theta, \tau) \varphi_n^*(u - \tfrac{1}{2}\tau\hbar) \varphi_m(u + \tfrac{1}{2}\tau\hbar)\, d\theta\, du\, d\tau. \tag{3.3}$$

In general the phase-space eigenfunctions will not form a complete orthogonal set. But if $|f| = 1$, the h_{nm}'s do form such a set in the space of q and p. It is straightforward to show that

$$\iint h_{nm}(q, p) h_{n'm'}^*(q, p)\, dq\, dp = (1/2\pi\hbar)\, \delta_{nn'} \delta_{mm'}, \tag{3.4}$$

$$\sum_{nm} h_{nm}(q, p) h_{nm}^*(q', p') = (1/2\pi\hbar)\, \delta(q - q')\delta(p - p'), \tag{3.5}$$

$$\iint h_{nm}(q, p)\, dp\, dq = \delta_{nm}, \tag{3.6}$$

$$\sum_n h_{nn}(q, p) = 1/2\pi\hbar. \tag{3.7}$$

4. MIXTURES

Up to now we have considered pure states given by the wavefunction ψ. In the case of a mixture the distribution function becomes

$$F(q, p, t; f) = \sum_{k=0}^{\infty} P_k F^{(k)}(q, p, t; f), \tag{4.1}$$

where P_k is the probability of having $\psi^{(k)}$, and $F^{(k)}$ is obtained from (1.5) with $\psi^{(k)}$ instead of ψ. If each $\psi^{(k)}$ is expanded in a complete orthonormal set φ_n then (4.1) becomes

$$F(q, p, t; f) = \sum_{knm} P_k a_n^{*(k)} a_m^{(k)} h_{nm}(q, p; f) \tag{4.2}$$

$$= \sum_{nm} \rho_{mn} h_{nm}(q, p; f), \tag{4.3}$$

where ρ_{nm} is the density matrix.

A necessary and sufficient condition for a given F to describe a pure state is

$$F(q, p) = \iiiint F(q', p') F(q'', p'') \times g(q' - q, q'' - q, p' - p, p'' - p)\, dq'\, dq''\, dp'\, dp'', \tag{4.4}$$

where

$$g = \frac{\hbar}{8\pi^3} \iiiint e^{i\theta(q''-q) + i\theta'(q'-q) + i\tau(p''-p) + i\tau'(p'-p)} \times e^{-\frac{1}{2}i\theta\tau'\hbar + \frac{1}{2}i\theta'\tau\hbar} f^*(\theta, \tau) f^*(\theta', \tau') \times f(\theta + \theta', \tau + \tau')\, d\theta\, d\theta'\, d\tau\, d\tau'. \tag{4.5}$$

To show that (4.4) is necessary, we use the well-known fact that

$$\rho^2 = \rho \tag{4.6}$$

is a necessary and sufficient condition for the existence of a pure state. From (4.3) and (3.4) we have

$$\rho_{mn} = 2\pi\hbar \iint F(q, p) h_{nm}^*(q, p)\, dq\, dp. \tag{4.7}$$

Substituting (4.7) in (4.6)

$$\iint F(q, p) h_{nm}^*(q, p)\, dq\, dp = 2\pi\hbar \sum_k \int F(q, p) \times F(q', p') h_{km}^*(q, p) h_{nk}^*(q', p')\, dq\, dq'\, dp\, dp'. \tag{4.8}$$

Multiplying by $h_{nm}(q'', p'')$ and summing over n and m we obtain, using (3.5),

$$F(q, p) = (2\pi\hbar)^2 \sum_{knm} \int F(q', p') F(q'', p'') \times h_{km}^*(q'', p'') h_{nk}^*(q'p') h_{nm}(q, p)\, dq'\, dq''\, dp'\, dp''.$$

But some algebra yields

$$(2\pi\hbar)^2 \sum_{knm} h_{km}^*(q'', p'') h_{nk}^*(q'p') h_{nm}(q, p) = g(q' - q, q'' - q, p' - p, p'' - p) \tag{4.9}$$

and therefore the necessity is established. To show that (4.4) is also sufficient, we write

$$F(q, p) = \sum_{nm} \rho_{mn} h_{nm}(q, p) \tag{4.10}$$

and substitute in (4.4)

$$\sum_{nm} \rho_{mn} h_{nm}(q, p) = \sum_{nmn'm'} \int \rho_{mn} \rho_{m'n'} h_{nm}(q'', p'') h_{n'm'}(q'p') \times g(q' - q, q'' - q, p' - p, p'' - p)\, dq'\, dq''\, dp'\, dp''. \tag{4.11}$$

But

$$\int h_{nm}(q'', p'')h_{n'm'}(q', p')g\, dq'\, dq''\, dp'\, dp'' = \delta_{nm'}h_{n'm}(q, p). \tag{4.12}$$

Thus

$$\sum_{nm} \rho_{mn}h_{nm}(q, p) = \sum_{nmn'm'} \rho_{mn}\rho_{m'n'}\delta_{nm'}h_{n'm}(q, p) = \sum_{mnn'} \rho_{mn}\rho_{nn'}h_{n'm}(q, p)$$

or

$$\rho_{mn} = \sum_{n'} \rho_{mn'}\rho_{n'n} = \rho^2_{mn},$$

which is a sufficient condition for a pure state.

5. DYNAMICS

A direct but somewhat lengthy calculation gives

$$\frac{\partial F(q, p, t; f)}{\partial t} = \frac{2}{\hbar} f^{-1}\left(i\frac{\partial}{\partial q_F}, i\frac{\partial}{\partial p_F}\right) f\left(-i\frac{\partial}{\partial q_H}, -i\frac{\partial}{\partial p_H}\right) f\left(i\frac{\partial}{\partial q_F} + i\frac{\partial}{\partial q_H}, i\frac{\partial}{\partial p_F} + i\frac{\partial}{\partial q_H}\right)$$

$$\times \sin\frac{\hbar}{2}\left[\frac{\partial}{\partial p_F}\frac{\partial}{\partial q_H} - \frac{\partial}{\partial p_H}\frac{\partial}{\partial q_F}\right]H(q, p)F(q, p, t; f) + \frac{\dot{f}\left(-i\dfrac{\partial}{\partial q_F}, -i\dfrac{\partial}{\partial p_F}\right)}{f\left(-i\dfrac{\partial}{\partial q_F}, -i\dfrac{\partial}{\partial p_F}\right)} F(q, p, t) \tag{5.1}$$

for the equation of motion of the generalized phase-space distribution. $H(q, p)$ is the classical Hamiltonian. $\partial/\partial q_H$, $\partial/\partial p_H$ operates on H only and $\partial/\partial q_F$, $\partial/\partial p_F$ on F. (5.1) reduces to the classical equation of motion if f is taken to be a function of $\hbar$ such that

$$\lim_{\hbar\to 0} f = 1$$

and

$$\lim_{\hbar\to 0} \dot{f} = 0.$$

We shall now derive the temporal transformation functions for the characteristic and distribution function in terms of the quantum mechanical Green's function. Let

$$M(\theta, \tau, t) = \iint L(\theta, \tau, t \mid \theta', \tau', t') \times M(\theta', \tau', t)\, d\theta'\, d\tau', \tag{5.2}$$

$$F(q, p, t) = \iint K(q, p, t \mid q', p', t') \times F(q', p', t')\, dq'\, dp', \tag{5.3}$$

$$\psi(q, t) = \int G(q, t \mid q', t')\psi(q', t')\, dq'. \tag{5.4}$$

From (1.5) and (2.1) we obtain for the characteristic function

$$M(\theta, \tau, t) = f(\theta, \tau) \int \psi^*(u - \tfrac{1}{2}\tau\hbar, t)e^{i\theta u} \times \psi(u + \tfrac{1}{2}\tau\hbar, t)\, du. \tag{5.5}$$

Inserting (5.4) into (5.5)

$$M(\theta, \tau, t) = f(\theta, \tau, t) \iiint e^{i\theta u}G^*(u - \tfrac{1}{2}\tau\hbar, t \mid q', t') \times G(u + \tfrac{1}{2}\tau\hbar, t \mid q, t')\psi^*(q', t')\psi(q, t)\, du\, dq\, dq'. \tag{5.6}$$

Making the substitution

$$q = u' + \tfrac{1}{2}\tau'\hbar,$$
$$q' = u' - \tfrac{1}{2}\tau'\hbar,$$

and inserting

$$1 = \frac{1}{2\pi}\iint e^{-i\theta'(u'-u'')}\, d\theta'\, du'',$$

we find that

$$M(\theta, \tau, t) = \frac{\hbar}{2\pi} f(\theta, \tau, t) \iiiint e^{i\theta u - i\theta' u'} \times G(u + \tfrac{1}{2}\tau\hbar, t \mid u' + \tfrac{1}{2}\tau'\hbar, t') \times G^*(u - \tfrac{1}{2}\tau\hbar, t \mid u' - \tfrac{1}{2}\tau'\hbar, t') \times [M(\theta', \tau', t)/f(\theta', \tau', t')]\, d\theta'\, d\tau'\, du\, du'. \tag{5.7}$$

Comparing with (5.2) we have

$$L(\theta, \tau, t \mid \theta'\tau', t') = \frac{\hbar}{2\pi}\frac{f(\theta, \tau, t)}{f(\theta', \tau', t')} \iint e^{i\theta u - i\theta' u'} \times G(u + \tfrac{1}{2}\tau\hbar, t \mid u' + \tfrac{1}{2}\tau'\hbar, t') \times G^*(u - \tfrac{1}{2}\tau\hbar, t \mid u' - \tfrac{1}{2}\tau'\hbar, t')\, du\, du'. \tag{5.8}$$

A similar procedure applied to (5.3) gives

$$K(q, p, t \mid q', p', t') = \frac{1}{4\pi^2} \iiiint L(\theta, \tau, t \mid \theta', \tau', t') \times e^{i\theta' q' + i\tau' p' - i\theta q - i\tau p} \, d\theta \, d\tau \, d\theta' \, d\tau'. \quad (5.9)$$

6. CONCLUSION

In conclusion we would like to mention some general features of the phase-space formulation. As mentioned in the introduction, the Wigner distributoin has been widely applied. It may be of some interest to repeat some of these calculations using (1.5) to determine how sensitive the results are to the choice of f.

It is commonly held that the uncertainty principle by itself precludes the possibility of the existence of a joint distribution of position and momentum. However, this is not so. For example, the choice

$$f(\theta, \tau, t) = \frac{\iint |\psi(q)|^2 \, |\phi(p)|^2 \, e^{i\theta q + i\tau p} \, dq \, dp}{\int \psi^*(u - \frac{1}{2}\tau\hbar) \, e^{i\theta u} \psi(u + \frac{1}{2}\tau\hbar) \, du} \quad (6.1)$$

leads to

$$F(q, p, t) = |\psi(q)|^2 \, |\phi(p)|^2, \quad (6.2)$$

which is certainly a well-defined joint distribution, and from which the uncertainty principle follows in the usual manner. The reason why a true joint distribution cannot be defined is because no choice of f yields a distribution which gives the correct quantum mechanical expectation values for all observables when calculated through phase-space integration. That is, no f exists such that, if the correspondence of quantum to classical variables derived in Sec. 2 is

$$g(q, p) \rightarrow \mathbf{G} \quad (6.3)$$

for some f, then also

$$H(g(q, p)) \rightarrow H(\mathbf{G}) \quad (6.4)$$

for the same f, where H is any function.

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NOTICE

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