Nuclear Fusion and Plasma Physics **EPFL**
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Solutions to Problem Set 3

Exercise 1 - Plasma production

a) The definition of **relative degree of ionization** is

$$\alpha = \frac{n_e}{n_e + n_{Ar}}$$

where $n_{Ar} = N_{Ar}/V$ is the density of neutral Argon atoms (number of Ar atoms per m^3). To evaluate n_{Ar} we can use the ideal gas law:

$$p_{Ar} = n_{Ar}\, k_B\, T_{Ar} \tag{1}$$

where p_{Ar} is the pressure of Argon inside the vacuum chamber in Pascal, $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann constant and T_{Ar} is the temperature of the Argon gas in Kelvin (normally assumed to be at room temperature - 298 K). Inverting this equation for n_{Ar} one finds that

$$n_{Ar} = \frac{p_{Ar}}{k_B\, T_{Ar}} \tag{2}$$

In order to use this expression one needs to convert the pressure given in Torr to Pascal using 760 Torr $= 1.01 \times 10^5$ Pa. Then $p_{Ar} = 1.33 \times 10^{-2}$ Pa. Thus, the neutral Argon number density n_{Ar} is

$$n_{Ar} = 3.23 \times 10^{18}\, \mathrm{m}^{-3} \tag{3}$$

The degree of ionization with $n_e = 1 \times 10^{16}\ \mathrm{m}^{-3}$ and $n_{Ar} = 3.23 \times 10^{18}\, \mathrm{m}^{-3}$ is

$$\alpha = \frac{n_e}{\underbrace{n_e + n_{Ar}}_{\approx n_{Ar}}} = \frac{1 \times 10^{16}}{1 \times 10^{16} + 3.23 \times 10^{18}} \approx 3.08 \times 10^{-3}$$

b) The electron-neutral **collision frequency** is

$$\nu_{en} = n_{Ar}\,\sigma_n\,v_{rel}$$

where v_{rel} is the relative velocity between electrons and neutrals and $\sigma_n = 10^3\,\pi\,a_0^2$ is the collision cross-section. Since $m_e \ll m_{Ar}$ and $T_e \gg T_0$, we can assume $v_{rel} \simeq v_e$.

In general, ν_{en} is a function of the electron velocity and, implicitly, $\sigma_n = \sigma_n(v_e)$. In our problem, we can consider σ_n constant and a typical velocity of the electrons equal to their thermal velocity $v_{the} = \sqrt{\frac{e\,T_e}{m_e}}$.

Plugging these numbers in the expression above we find:

$$\nu_{en} = 3.23 \times 10^{18}\ \mathrm{m}^{-3} \cdot 10^3\,\pi\,\underbrace{(5.29 \times 10^{-11})^2}_{a_0^2}\ \mathrm{m}^2 \cdot \sqrt{\frac{e\,T_e}{m_e}}\,\frac{\mathrm{m}}{\mathrm{s}} \approx 2.06 \times 10^7\,\mathrm{s}^{-1}$$

c) Can we consider this gas to be a **plasma?**

- The Debye length is:

$$\lambda_D = \sqrt{\frac{\varepsilon_0\,T_e}{e^2\,n_e}} \approx 7430\sqrt{\frac{T_e[\mathrm{eV}]}{n_e\,[\mathrm{m}^{-3}]}} = 7430\sqrt{\frac{3}{10^{16}}}\,\mathrm{m} = 0.13\ \mathrm{mm}$$

  The ionized gas is confined in a container of dimension $L_p \approx 0.5\,\mathrm{m} \gg 0.13\,\mathrm{mm}$. We see then that $L_p \gg \lambda_D$ is required for a plasma.

- $N_D = \frac{4}{3}\pi\,\lambda_D^3\,n_e \approx 9.2 \times 10^4 \gg 1$, so the condition of the plasma parameter $g = N_D^{-1} \ll 1$ is verified.

- To see dynamic collective effects in a plasma (oscillations at the frequency ω_p), we need ω_p to be much larger than the collision frequency:

$$\omega_p = \sqrt{\frac{e^2\,n_e}{m_e\,\varepsilon_0}} \approx 18\pi\,\sqrt{n_e\,[\mathrm{m}^{-3}]}\ \ \mathrm{rad/s} = 18\pi\,\sqrt{10^{16}}\ \mathrm{rad/s} = 5.7 \times 10^9\ \mathrm{rad/s}$$

  To compare ω_p with ν_{en} we need to convert it in s^{-1}:

$$f_p = \frac{\omega_p}{2\pi} \approx 0.9 \times 10^9\ \mathrm{s}^{-1} > \nu_{en} = 2.06 \times 10^7\,\mathrm{s}^{-1}$$

We can therefore conclude that this ionized gas *is* a plasma.

Exercise 2 - Mirror effect

a) In an ideal solenoid, the field lines would be straight and uniform through the ring-shaped conductors, creating a consistent magnetic field within the solenoid. However, due to the finite distance between the two coils, the field lines will diverge after passing through the first ring, spreading out slightly in the space between the coils, and then reconverging as they enter the second ring. This divergence creates a weaker magnetic field in the central region and a stronger field near the coils. This divergence and reconvergence of the field lines result in a non-uniform magnetic field between the coils, which can affect the overall field strength and distribution.

b) The field will be strongest at the location of the coils, and weaker in the middle. This variation in field strength is due to the concentration of magnetic field lines near the coils. This is because the magnetic field lines are more concentrated near the coils, leading to a higher field strength in these regions. Far to the left and right of the coils, the field will decay approximately as $1/r^3$, similarly as for a magnetic dipole.

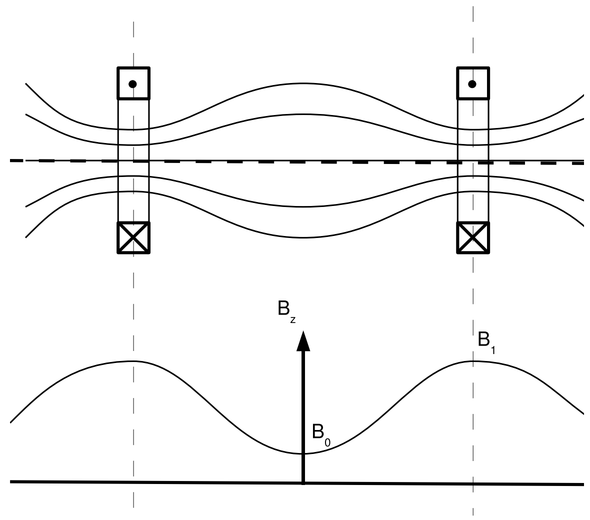

Figure 1: Field lines and field strength in a magnetic mirror.

c) A particle with velocity only along the axis will feel no force because it moves parallel to the magnetic field. It will continue undisturbed along its path. The particle's motion is unaffected because the magnetic field exerts no force on charges moving parallel to it.

d) Denote the minimum magnetic field strength halfway between the coils as B_0 and the maximum field strength at each coil as B_1. Conservation of kinetic energy gives

$$\frac{1}{2}mv_{||,0}^2 + \frac{1}{2}mv_{\perp,0}^2 = \frac{1}{2}mv_{||,1}^2 + \frac{1}{2}mv_{\perp,1}^2$$

Conservation of the adiabatic invariant, which states that the magnetic moment is

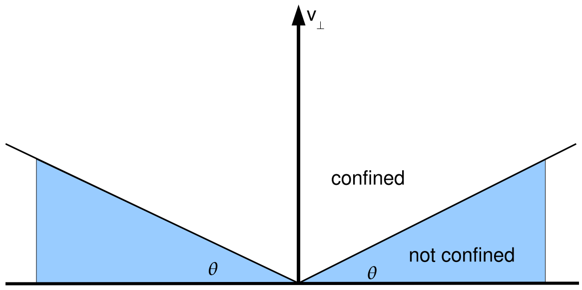

Figure 2: Loss cone of a particle in a magnetic mirror.

conserved in a slowly varying magnetic field, gives

$$\frac{mv_{\perp,0}^2}{B_0} = \frac{mv_{\perp,1}^2}{B_1}$$

Using this result, one can substitute for $v_{\perp,1}^2$ to obtain

$$\frac{1}{2}mv_{||,0}^2 + \frac{1}{2}mv_{\perp,0}^2\left(1 - \frac{B_1}{B_0}\right) = \frac{1}{2}mv_{||,1}^2$$

Since $B_1 > B_0$, the second term on the left-hand side is negative. If B_1 is large enough, then this term will cancel the first term, meaning that $v_{||,1}$ can become zero. This indicates that the parallel component of the velocity is completely converted into perpendicular velocity, meaning that the particle can be reflected back, which is the basis of magnetic mirror confinement.

e) If a particle is reflected, there must be a point on its trajectory where $v_{||,1} = 0$. Rearranging the expression found in (d), we get

$$\frac{1}{2}mv_{||,0}^2 + \frac{1}{2}mv_{\perp,0}^2 = \frac{B_1}{B_0}\frac{1}{2}mv_{\perp,0}^2 \quad (4)$$

$$\frac{v_{||,0}^2 + v_{\perp,0}^2}{v_{\perp,0}^2} = \frac{B_1}{B_0} \quad (5)$$

$$\sin^2\theta_c = \frac{B_0}{B_1} \quad (6)$$

Here, θ_c is the critical angle on the $(v_{||,0}, v_{\perp,0})$ plane (see Fig. 2). The zone where $\theta < \theta_c$, corresponding to small perpendicular velocities, is referred to as the *loss cone* of the velocity distribution. Particles with velocity components within this loss cone at the midplane will not be trapped in the magnetic mirror and will escape the confinement.

The loss cone is one of the fundamental reasons why achieving fusion based on magnetic mirrors is difficult. For effective confinement, we would like particles to have a high perpendicular velocity, which helps keep them within the mirror. However, due to collisions within the plasma, particles will gradually acquire a parallel velocity component as well, pushing them into the loss cone and allowing them to escape from the mirror confinement.

Exercise 3 - Confinement by a toroidal field

a) Ampère's law in integral form reads

$$\oint_C \mathbf{B} \cdot \mathbf{d}\ell = \mu_0 \iint_S \mathbf{j} \cdot \mathbf{dS} = \mu_0 I$$

This is valid for any contour containing the current-carrying wire. Due to the cylindrical symmetry, if we choose the contour as a circle centered at the wire, B is constant along the integration path. Therefore,

$$B \oint_C d\ell = 2\pi r B_\theta = \mu_0 I$$
$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

Clearly, the field strength decreases as $1/r$.

Using Ampère's law in differential form (in cylindrical coordinates), we get a similar result:

$$(\nabla \times B)_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right] = 0$$

We immediately remove terms involving $\partial/\partial\theta$ because of symmetry.

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = 0$$
$$r B_\theta = k$$
$$B_\theta = \frac{k}{r}$$

with k being the appropriate constant of integration. In this case, as well, we see that the field strength decreases as $1/r$.

The gradient of the magnetic field strength is easy to compute since the only component is B_θ, which depends only on r.

$$\begin{aligned}\nabla B &= \frac{d}{dr} B_\theta(r)\,\hat{\mathbf{r}} \\ &= \frac{d}{dr}\left(\frac{\mu_0 I}{2\pi r}\right)\hat{\mathbf{r}} \\ &= -\frac{\mu_0 I}{2\pi r^2}\,\hat{\mathbf{r}}\end{aligned}$$

As expected, the gradient is oriented in the $-\hat{\mathbf{r}}$ direction.

b) We can now evaluate the direction of the various drifts. The ∇B drift is in the direction of $\mp B \times \nabla B$ for electrons and ions, respectively. The curvature drift has the direction

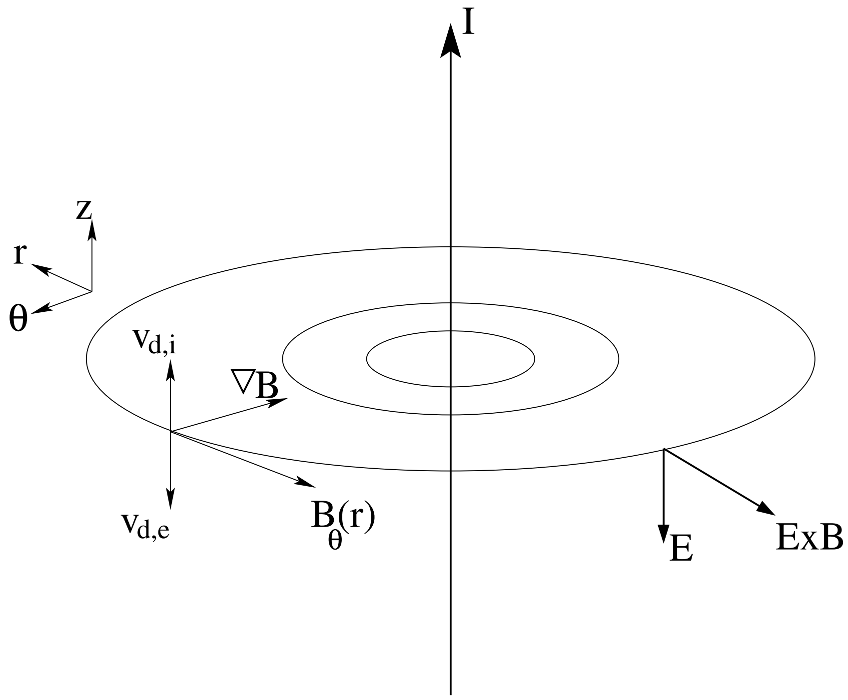

Figure 3: Toroidal B field created by a current-carrying wire. The ∇B-drift (which is opposite for electrons and ions) causes charge separation, which creates an electric field. The resulting $E \times B$ drift drives the bulk of the plasma outwards.

$\mp R_c \times B$ for electrons and ions, respectively. Since ∇B and R_c are opposite, both drifts will have the same effect: they will cause ions to drift upwards and electrons to drift downwards.

This charge separation will result in an electric field in the $-\hat{\mathbf{z}}$ direction: $\mathbf{E} = -E\,\hat{\mathbf{z}}$. The presence of this electric field leads to the $E \times B$ drift, which is oriented in the outward radial direction for both ions and electrons. Consequently, this outward drift pushes the bulk of the plasma away from the center, causing the plasma to escape. This is the fundamental reason why it is not possible to confine a plasma in a simple toroidal field.

Note: In a Tokamak, this problem is solved by driving a current through the plasma. The resulting poloidal magnetic field will add to the toroidal field, producing helically twisted field lines which periodically visit both the top and bottom parts of the plasma. This provides a path for the particles to counteract the charge separation, thus "short-circuiting" this instability.