Solution 1
19/09/2013

One dimensional random walk

Each realization of the RW can be represented as a sequence of pluses and minuses, where + stands for a forward step and − for a backward one. For example, the sequence $(+-+++-++--+-)$ represents a RW with a total of $N = 12$ steps, where the walker started performing a forward step, followed by a backward one, still followed by three forward steps, and so on. It is clear that the final position of the walker does not depend on the order of the steps performed in the sequence, but only on the difference between the number n_+ of forward steps and the n_- backward ones. Within this framework, the final position is given by $n = n_+ - n_-$, while the total number of steps is $n_+ + n_- = N$. From these equations it is easy to show that

$$n_+ = \frac{N+n}{2} \qquad n_- = \frac{N-n}{2} \tag{1}$$

Now, the probability to realize a particular sequence is given by $p_+^{n_+} p_-^{n_-}$. Therefore, in order to find the probability $P(n, N)$ to be at position n we just have to multiply such a probability by the number of sequences compatible with the given value of n. In order to define a sequence, it is sufficient to give the positions of e.g. the various +. Now, since the number of pluses is n_+ and the avalaible spots are N, we have $\binom{N}{n_+}$ possible sequences. The final result is thus

$$P(n, N) = p_+^{n_+} p_-^{n_-} \frac{N!}{n_+! n_-!} = p_+^{n_+} p_-^{n_-} \frac{N!}{\left(\frac{N+n}{2}\right)! \left(\frac{N-n}{2}\right)!} \tag{2}$$

1. In the unbiased case $p_+ = p_- = 1/2$. Remembering that $n_+ + n- = N$, we thus have $p_+^{n_+} p_-^{n_-} = 1/2^N$. By substituting in the previous equation we thus obtain

$$P_0(n, N) = \frac{1}{2^N} \frac{N!}{\left(\frac{N+n}{2}\right)! \left(\frac{N-n}{2}\right)!} \tag{3}$$

2. Starting from the formula obtained in 1. and making use of the Stirling approximation we get

$$\begin{aligned} P_0(n, N) &= \frac{1}{2^N} \frac{N!}{(\frac{N+n}{2})!(\frac{N-n}{2})!} \\ &\approx \frac{1}{2^N} \frac{\sqrt{2\pi N} N^N e^{-N}}{\sqrt{2\pi \frac{N+n}{2}} \sqrt{2\pi \frac{N-n}{2}} \left(\frac{N+n}{2}\right)^{\frac{N+n}{2}} \left(\frac{N-n}{2}\right)^{\frac{N-n}{2}} e^{-\frac{N+n}{2}} e^{-\frac{N-n}{2}}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{2^{1+\frac{N+n}{2}+\frac{N-n}{2}}}{2^N} \frac{N^{N+\frac{1}{2}}}{(N+n)^{\frac{N+n}{2}+\frac{1}{2}} (N-n)^{\frac{N-n}{2}+\frac{1}{2}}} \\ &= \frac{2}{\sqrt{2\pi N}} \frac{1}{(1+\frac{n}{N})^{\frac{N+n}{2}+\frac{1}{2}} (1-\frac{n}{N})^{\frac{N-n}{2}+\frac{1}{2}}} \, . \end{aligned}$$

where in the last step we factorized N in each bracket.

Now, by taking the logarithm of first and last side in the previous equation and making use of the Taylor expansion $\ln(1 \pm \frac{n}{N}) \approx \pm\frac{n}{N} - \frac{n^2}{2N^2}$ to the *second* order[1], we obtain

$$\begin{aligned}
\ln P_0(n,N) &= \ln\frac{2}{\sqrt{2\pi N}} - \left(\frac{N}{2}+\frac{n}{2}+\frac{1}{2}\right)\ln\left(1+\frac{n}{N}\right) - \left(\frac{N}{2}-\frac{n}{2}+\frac{1}{2}\right)\ln\left(1-\frac{n}{N}\right) \\
&\approx \ln\frac{2}{\sqrt{2\pi N}} - \left(\frac{N}{2}+\frac{n}{2}+\frac{1}{2}\right)\left(\frac{n}{N}-\frac{n^2}{2N^2}\right) + \left(\frac{N}{2}-\frac{n}{2}+\frac{1}{2}\right)\left(\frac{n}{N}+\frac{n^2}{2N^2}\right) \\
&\approx \ln\frac{2}{\sqrt{2\pi N}} - \frac{n^2}{2N}\ ,
\end{aligned}$$

which gives:

$$P_0(n,N) \approx \frac{2}{\sqrt{2\pi N}} e^{-\frac{n^2}{2N}}\ .$$

Nevertheless, the previous formula has some problems. Indeed, by calculating the integral we obtain $\int_{-\infty}^{+\infty} P_0(n,N)dn = 2$ instead of 1. Such a "defect" in the normalization arises from a little inaccuracy in our derivation. Indeed, if the walker performs an even number of steps N, he will be able to reach only even positions n. The exact probability for N even should thus be written:

$$P_e(n,N) = \begin{cases} P_0(n,N) & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

and viceversa for N odd.

Because of such a detail, when approximating sums by integrals as we did calculating the normalization we turn out to count twice each probability. The correct result has thus to be obtained by dividing by two the formula we computed

$$P_0(n,N) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{n^2}{2N}} \tag{4}$$

3. Starting again from eqn (2) and performing the substitutions $p_+ = (1+\epsilon)/2$ and $p_- = (1-\epsilon)/2$ we get

$$\begin{aligned}
P_\epsilon(n,N) &= \left(\frac{1+\epsilon}{2}\right)^{\frac{N+n}{2}} \left(\frac{1-\epsilon}{2}\right)^{\frac{N-n}{2}} \frac{N!}{\left(\frac{N+n}{2}\right)!\left(\frac{N-n}{2}\right)!} = \\
&= \frac{1}{2^N}\frac{N!}{\left(\frac{N+n}{2}\right)!\left(\frac{N-n}{2}\right)!}(1+\epsilon)^{\frac{N+n}{2}}(1-\epsilon)^{\frac{N-n}{2}} = \\
&= P_0(n,N)(1+\epsilon)^{\frac{N+n}{2}}(1-\epsilon)^{\frac{N-n}{2}}
\end{aligned}$$

By taking the logarithm, making use of eqn (4) and expanding $\log(1\pm\epsilon)$ up to the second order, we obtain

$$\begin{aligned}
\log P_\epsilon(n,N) &= \log\frac{1}{\sqrt{2\pi N}} - \frac{n^2}{2N} + \frac{N+n}{2}\left(\epsilon - \frac{\epsilon^2}{2}\right) + \frac{N-n}{2}\left(-\epsilon-\frac{\epsilon^2}{2}\right) = \\
&= \log\frac{1}{\sqrt{2\pi N}} - \frac{n^2}{2N} + n\epsilon - \frac{N\epsilon^2}{2} = \\
&= \log\frac{1}{\sqrt{2\pi N}} - \left(\frac{n}{\sqrt{2N}} - \sqrt{\frac{N}{2}}\epsilon\right)^2 = \\
&= \log\frac{1}{\sqrt{2\pi N}} - \frac{1}{2N}(n-N\epsilon)^2
\end{aligned}$$

[1] As we will see, because of prefactors the terms coming from the first two orders are of comparable size. Therefore, it would be a mistake to stop just at the first order (however, it is possible to show that the subsequent terms in the expansion lead to negligible terms).

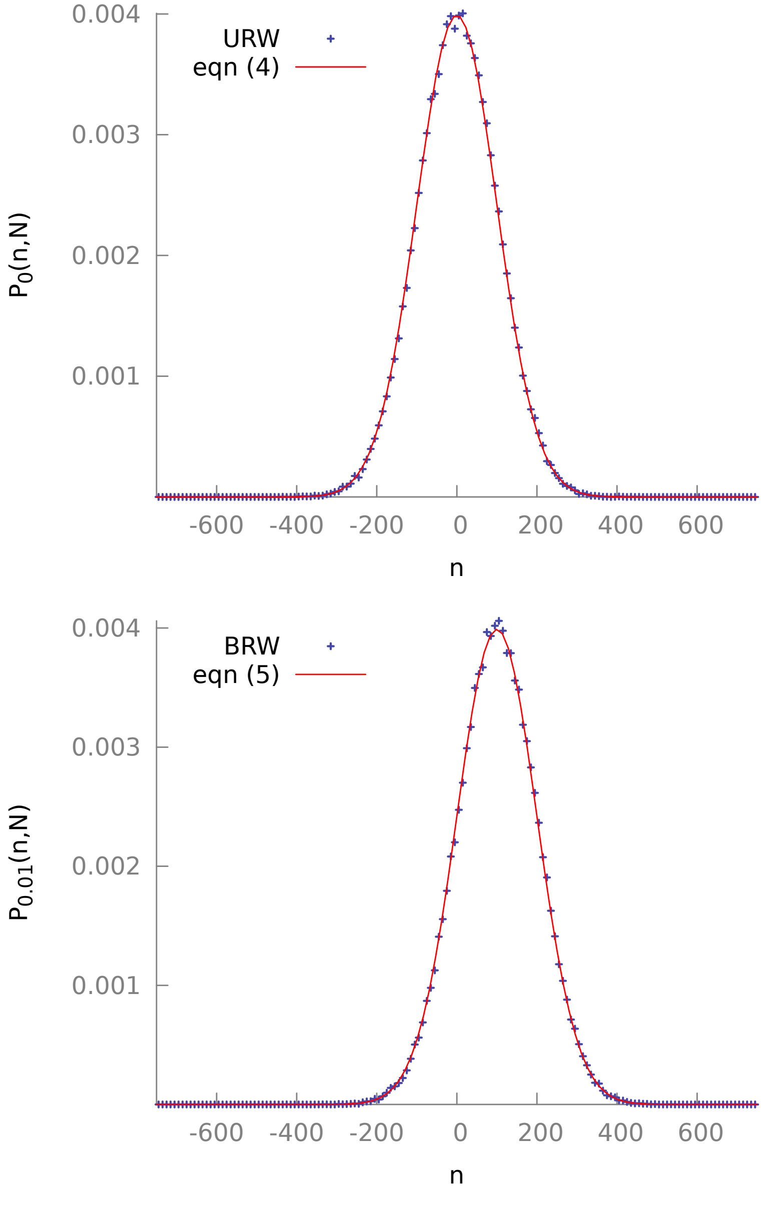

Figure 1: Top: simulation results (blue points) and analytical formula (red curve, see eqn (4)) for the probability distribution of an Unbiased Random Walk (URW) with $N = 10^4$. Bottom: results for a Biased Random Walk (BRW) with $N = 10^4$ and $\epsilon = 0.01$. In both cases the numerical results shown are averages taken over 10^5 independent realizations.

Therefore, for the biased RW we finally find

$$P_\epsilon(n,N) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{(n-N\epsilon)^2}{2N}} \tag{5}$$

which is a Gaussian function with average $N\epsilon$ and variance N. A comparison with eqn (4) shows immediately that the introduction of a bias introduces a shift in the average value (as one could have intuitively guessed without calculations), but it does not change the distribution function and the size of fluctuations. Moreover, putting $\epsilon = 0$ one retrieves eqn (4), as it should be. In fig.1 we show the comparison between simulation data over 10^5 realizations for both Unbiased (top) and Biased (bottom,$\epsilon = 0.01$) RWs (denoted in the picture as URW and BRW respectively) with $N = 10^4$, together with the analytical formulas (4) and (5).

4. At each step of a RW, we have to decide whether to go a step forward (+1) or backward (−1), with probabilities $p_+ \equiv p$ and $p_- = 1 - p$ respectively. Therefore, if we introduce the random variable "step" S, such a variable takes value +1 with probability p and −1 with probability $1-p$. Within this framework, the final position n can be thought of as the sum of the N independent variables S_i, each distributed according to the previous rules. In the limit of large N, according to the Central Limit Theorem (CLT) we can already conclude that the variable n will be distributed according to a Gaussian. We now proceed in calculating its mean and variance, which unambigouosly define it. In order to do so, we will use the results of the CLT, so that we first need to calculate the average μ and variance σ^2 of the "step" distribution S. The mean is given by

$$\mu = p_+ \cdot (+1) + p_- \cdot (-1) = p - (1-p) = 2p - 1,$$

while the variance is

$$\begin{aligned} \sigma^2 &= p_+ \cdot (+1-\mu)^2 + p_- \cdot (-1-\mu)^2 = \\ &= p(2-2p)^2 + (1-p)(-2p)^2 = \\ &= 4p(1-p)^2 + 4p^2(1-p) = \\ &= 4p(1-p)(1-p+p) = \\ &= 4p(1-p) \end{aligned}$$

Now, since $p = p_+ = \frac{1}{2}(1+\epsilon)$, the previous results become respectively

$$\mu = \epsilon$$

and

$$\sigma^2 = (1+\epsilon)(1-\epsilon) = 1-\epsilon^2$$

Therefore, by applying the CLT we conclude that the final position n is distributed according to a Gaussian with mean $N\epsilon$ and variance $N(1-\epsilon^2)$:

$$P_\epsilon(n,N) = \frac{1}{\sqrt{2\pi N(1-\epsilon^2)}} e^{-\frac{(n-N\epsilon)^2}{2N(1-\epsilon^2)}} \tag{6}$$

We stress that this result is valid for any $|\epsilon| < 1$. Moreover, note that when $\epsilon \ll 1$ formula (5) is retrieved. Apart from shifting the average value to $N\epsilon$, the introduction of a bias gives less and less room for fluctuations, whose size decreases as $\sqrt{1-\epsilon^2}$[2]. In fig.2 we show the comparison between simulation results and eqn (6) for a strongly biased RW ($\epsilon = 0.9$). As expected, the agreement between numerical data and theoretical formula is excellent.

[2]What happens when we reach the limit case $\epsilon = 1$?

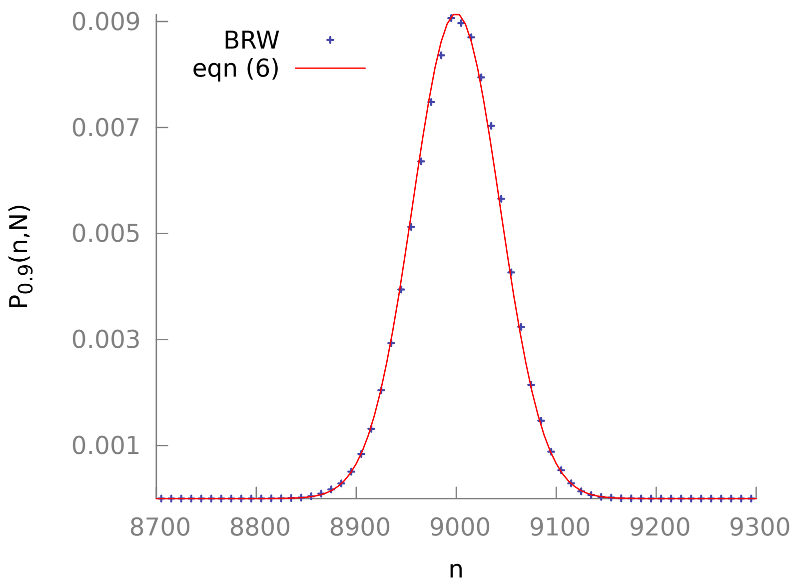

Figure 2: Simulation results (blue points) and analytical formula (red curve, see eqn (6)) for the probability distribution of a BRW with $N = 10^4$ and $\epsilon = 0.9$. The numerical results shown are averages taken over 10^5 independent realizations.