Solution 5

Use of the Generating Function: Waiting Time for the First Gain

1. If we interpret C_n as the position of a one dimensional random walker after n steps, then p_n corresponds to the probability of being for the first time one position to the right of the starting position after n step.
   Starting from $C_0 = 0$, after the first step, we have either $C_1 = 1$ with probability p, which implies $p_1 = p$, or $C_1 = -1$ with probability $1 - p$.

2. Assuming that in the first game Alex has lost his bet, at this point his payoff will be $C_1 = -1$. By the very definition of p_n, the probability that the payoff reaches the zero in other $n - 1$ steps (in order to have n steps in total) is simply given by p_{n-1}, since we basically "shifted" the origin to the -1 position. By considering that the first loss happens with probability $1 - p$, we thus find for the event "first return to zero" the probability $(1 - p)p_{n-1}$.

3. In order to have $C_n = 1$ for a given $n \geq 2$, we should first have $C_k = 0$ at least for one $k \in [2, n - 1]$ (the upper bound is n decreased by one because the walker has to perform a final step from 0 to +1 in the end). In particular, there will be a first passage time by zero. As we computed in the previous point, the probability that we re-reach the zero in k steps for the first time is exactly p_{k-1}. At this point, as a net result the walker is found exactly in the zero position after k steps, so that it will reach the +1 position for the first time with probability p_{n-k}. Summarizing, for given k and n, this sequence of events happens with probability

   $$(1 - p)p_{k-1}p_{n-k}$$

   For $n \geq 2$, the probability p_n will then be the sum of this events over $k = 1, \ldots, n - 1$. On the other hand, for $n = 1$ we now that $p_1 = p$, so that we can finally write

   $$p_n = p\delta_{n,1} + (1 - \delta_{n,1})(1 - p) \sum_{k=1}^{n-2} p_k p_{n-k-1} \tag{1}$$

   where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. Notice that $p_n = 0$ for all the even values of n (such terms were included in the previous sum just because of convenience).

4. By applying the definition of the generating function and making some computations, we get

   $$\begin{aligned} G(s) &= \sum_{n=1}^{\infty} p_n s^n = \\ &= \sum_{n=1}^{\infty} \left[p\delta_{n,1} + (1 - \delta_{n,1})(1 - p) \sum_{k=1}^{n-2} p_k p_{n-k-1} \right] s^n = \\ &= ps + (1 - p) \sum_{n=3}^{\infty} \sum_{k=1}^{n-2} p_k p_{n-k-1} s^{k+(n-k-1)+1} = \\ &= ps + (1 - p)s \sum_{k=1}^{\infty} \sum_{n=k+2}^{\infty} p_k p_{n-k-1} s^k s^{n-k-1} = \\ &= ps + (1 - p)s \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} p_k p_m s^k s^m = \end{aligned}$$

$$
\begin{aligned}
&= ps + (1-p)s \left(\sum_{k=1}^{\infty} p_k s^k \right)^2 = \\
&= ps + (1-p)s \left[G(s) \right]^2
\end{aligned}
$$

By comparing the first and last term we thus obtain the following equation for $G(s)$

$$
(1-p)s \left[G(s) \right]^2 - G(s) + ps = 0 \tag{2}
$$

By solving this equation, we find two solutions

$$
G_{\pm}(s) = \frac{1 \pm \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s}
$$

However, from the very definition of the generating function, it is easy to prove that $G'(0) = p_1 = p$. Making some calculations and remembering that for $x \to 0$ the Taylor expansion of the square root is $\sqrt{1+x} \simeq 1 + x/2$, we obtain that for $s \to 0$

$$
G'_{+}(s) \simeq -\frac{2p}{1 - 2p(1-p)s^2} - \frac{1 - p(1-p)s^2}{(1-p)s^2} \xrightarrow{s \to 0} -\infty \tag{3}
$$

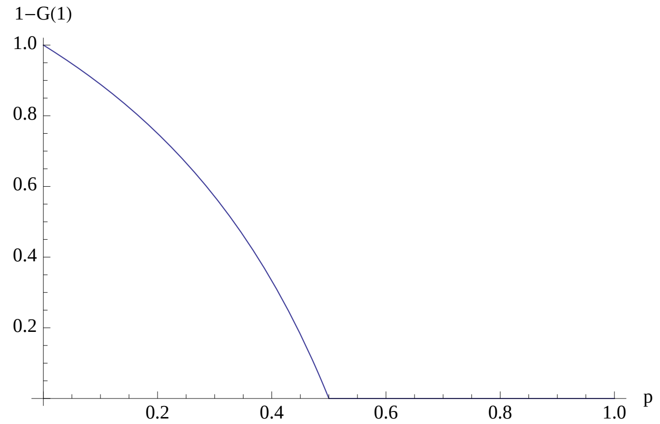
which makes this solution unacceptable. On the other hand

$$
G'_{-}(s) \simeq \frac{2p}{1 - 2p(1-p)s^2} - \frac{2p(1-p)s^2}{2(1-p)s^2} \xrightarrow{s \to 0} p \tag{4}
$$

as it should be. We thus conclude that G_- is the seeked generating function.

5. Note that the condition $G(1) = \sum_n p_n = 1$ does not hold for all the values of p: when $p < 1/2$, it is indeed possible that the random walker never reaches 1! In this section we propose to compute this *escape probability*. The probability that $C_n < 1 \; \forall n \geq 0$ is given by

$$
\begin{aligned}
1 - (p_0 + p_1 + p_2 + \ldots) = 1 - G(1) = 1 - \frac{1 - \sqrt{1 - 4p(1-p)}}{2(1-p)} \\
= \frac{1 - 2p + |1 - 2p|}{2(1-p)} = \begin{cases} 0 & \text{if } p \geq 1/2 \\ 1 - \dfrac{p}{1-p} & \text{otherwise} \end{cases} .
\end{aligned}
$$



The average number of trials to wait for it is $\langle n \rangle = G^{'}(1)$. Unfortunately, for $p = 1/2$, this gives

$$G^{'}(s) = \frac{\frac{s^2}{\sqrt{1-s^2}} - \left(1 - \sqrt{1 - s^2}\right)}{s^2} = \frac{1 - \sqrt{1 - s^2}}{s^2\sqrt{1 - s^2}} \xrightarrow{s \to 1} \infty \ .$$

In other words, although for $p = 1/2$ the walker will reach for sure the position $+1$, it will take him on average an infinite amount of time to do it! As a final remark, we note that, since for $p < 1/2$ there is a finite probability for the position $+1$ not to be reached, the formula $< n >= G'(1)$ does not hold in this case, since we have to have to weight the $n = \infty$ case with a non-zero probability (in other words, in this case $G'(1)$ measures the average expected time for all the successfull walks).