# Homework 5

## Use of the Generating Function: Waiting Time for the First Gain

Alex and Brad are gambling. Assume that Alex's probability to win a bet is $p$ and let $C_n$ be the number of his gains minus the number of his losses. Let $p_n$ be the probability that $C_n = 1$ for the *first time* after $n$ trials.

1. What is the value of $p_1$? Which condition has the first game to fulfill in order to have $C_n = 1$ for the first time at $n > 1$?
2. How would you compute the probability that, apart from the very beginning, $C_n = 0$ for the first time after $n > 1$ steps?
3. By making use of the results obtained in the previous points, show that $p_n$ can be written in terms of $p_{n-1}, \dots, p_1$ in a formula involving a *convolution* (see reminder below).
4. Show that the generating function $G(s)$ of $p_n$ is given by

$$G(s) = \frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s} \tag{1}$$

5. Which is the probability that $C_n$ remains negative forever?
6. Which is the expectation value of the number of trials before $C_n$ gets positive?
   **Hint:** use $G(s)$ and its properties.

## Reminder

- Given a random variable $\kappa$ with a probability distribution $p_k : \mathbb{N} \to [0, 1]$, its generating function $G(s) : \mathbb{C} \to \mathbb{C}$ is defined as

$$G(s) \equiv \langle s^\kappa \rangle = \sum_{k=1}^{\infty} p_k s^k \ .$$

  Note that $G(s)$ satisfies

$$p_k = \frac{1}{k!} \left. \frac{d^k G(s)}{ds^k} \right|_{s=0} \ ; \qquad \langle k^m \rangle = \left[ \left( s \frac{d}{ds} \right)^m G(s) \right]_{s=1} \ .$$

- The convolution of two sequences $\{a_k\}$ and $\{b_k\}$ is the sequence

$$\{c_r = a_0 b_r + a_1 b_{r-1} + ... + a_r b_0\} \ .$$