Homework 1
19/09/2013

One dimensional random walk

Consider a Random Walk (RW) on a 1 dimensional lattice: at each step the walker can decide whether to move forward to the next lattice site, or backward to the previous one. For simplicity, and without loss of generality, we can assume that the initial position of the walker is the origin and that the lattice spacing is equal to 1. Let p_+ and p_- be the probabilities of performing a forward or a backward step respectively.

1. In the case where $p_+ = p_- = 1/2$, compute the probability $P_0(n, N)$ that after N such steps the walker is found at position n on the lattice.

2. Show that in the limit $N \gg |n|$ the probability $P_0(n, N)$ can be written as

$$P_0(n, N) \approx \frac{1}{\sqrt{2\pi N}} e^{-\frac{n^2}{2N}} \tag{1}$$

   Hint: in the limit of large N, it can be shown that the so-called *Stirling approximation* holds

$$N! \approx \sqrt{2\pi N} N^N e^{-N} \ ; \tag{2}$$

3. Within the Stirling approximation, find the distribution $P_\epsilon(n, N)$ for a *biased* random walk[1] with $p_+ = \frac{1}{2}(1 + \epsilon)$ and $p_- = \frac{1}{2}(1 - \epsilon)$, where $|\epsilon| \ll 1$. Why do we need such a requirement on ϵ?

4. According to the so-called *Central Limit Theorem* (in its simplest form), the sum of N independent variables, each following the same probability distribution ρ, follows a Gaussian distribution. Moreover, if the mean and variance of ρ are μ and σ^2, then the resulting Gaussian will have average $N\mu$ and variance $N\sigma^2$. As an important remark, ρ does not need specific requirements (it could be itself a Gaussian or rather a strongly asymmetric distribution). Rederive the previous results by making use of such a theorem and find a formula valid for any arbitrary ϵ such that $|\epsilon| < 1$.

[1] i.e. a walk where $p_+ \neq p_-$.