Quantum Field Theory

Homeworks

Exercise 1

Given a Lorentz transformation $\Lambda : x \longrightarrow x' = \Lambda x$, the transformation of a scalar field at fixed coordinate x is $\phi(x) \longrightarrow \phi'(x)$. Since the field is scalar, it satisfies $\phi'(x') = \phi(x)$, or $\phi'(x) = \phi(\Lambda^{-1}x)$, which is the representation of the Lorentz transformation on functions. We can now expand for infinitesimal transformation:

$$\delta_0\phi(x) \equiv \phi'(x) - \phi(x) = \phi(\Lambda^{-1}x) - \phi(x) \simeq \phi(x - \omega x) - \phi(x) \simeq -\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x).$$

This variation has to be identified with the infinitesimal action of the Lorentz generators on scalar fields, namely

$$\delta_0\phi(x) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right]\phi(x) - \phi(x) \simeq -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x),$$

which proves that

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

is the representation of Lorentz generators on scalar fields. (Notice that the generators in this representation have been denoted as $L^{\mu\nu}$, not as $\mathcal{J}^{\mu\nu}$, since $\mathcal{J}^{\mu\nu}$ are the generators in the defining representation). It's now straightforward to obtain the Lorentz algebra:

$$\begin{aligned}
[L^{\mu\nu}, L^{\rho\sigma}] &= i^2\left[(x^\mu\partial^\nu - x^\nu\partial^\mu), (x^\rho\partial^\sigma - x^\sigma\partial^\rho)\right] \\
&= -\left[(x^\mu\eta^{\nu\rho}\partial^\sigma - x^\rho\eta^{\mu\sigma}\partial^\nu) + (x^\nu\eta^{\mu\sigma}\partial^\rho - x^\sigma\eta^{\nu\rho}\partial^\mu) - (x^\nu\eta^{\mu\rho}\partial^\sigma - x^\rho\eta^{\nu\sigma}\partial^\mu) - (x^\mu\eta^{\nu\sigma}\partial^\rho - x^\sigma\eta^{\mu\rho}\partial^\nu)\right] \\
&= -\left[\eta^{\nu\rho}(x^\mu\partial^\sigma - x^\sigma\partial^\mu) + \eta^{\mu\sigma}(x^\nu\partial^\rho - x^\rho\partial^\nu) + \eta^{\nu\sigma}(x^\rho\partial^\mu - x^\mu\partial^\rho) + \eta^{\mu\rho}(x^\sigma\partial^\nu - x^\nu\partial^\sigma)\right] \\
&= i\left[\eta^{\nu\rho}L^{\mu\sigma} + \eta^{\mu\sigma}L^{\nu\rho} + \eta^{\nu\sigma}L^{\rho\mu} + \eta^{\mu\rho}L^{\sigma\nu}\right].
\end{aligned}$$

Consider now translations $x \longrightarrow x' = x + a$. The transformation of a scalar field at fixed coordinate is $\phi(x) \longrightarrow \phi'(x)$, and since $\phi'(x') \equiv \phi'(x + a) = \phi(x)$, then $\phi'(x) = \phi(x - a)$. Identifying

$$\phi'(x) = \exp[ia^\mu P_\mu]\phi(x) = \phi(x - a) = \exp[-a^\mu\partial_\mu]\phi(x),$$

one immediately finds that the representation of the generators of translations on fields is given by

$$P_\mu = i\partial_\mu.$$

Using this explicit representation, the commutators are

$$\begin{aligned}
[P^\mu, P^\nu] &= i^2\left[\partial^\mu, \partial^\nu\right] = 0, \\
[P^\mu, L^{\rho\sigma}] &= i^2\left[\partial^\mu, x^\rho\partial^\sigma - x^\sigma\partial^\rho\right] = -(\eta^{\mu\rho}\partial^\sigma - \eta^{\mu\sigma}\partial^\rho) = i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho).
\end{aligned}$$

These relations are general since the structure constants of course do not depend on the representation used to compute the commutators. The above relations define thus the Poincaré algebra in *any* representation. Summarizing, the commutation relations between Poincaré generators are:

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i\left[\eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\nu\sigma}J^{\rho\mu} + \eta^{\mu\rho}\mathcal{J}^{\sigma\nu}\right], \\
[P^\mu, \mathcal{J}^{\rho\sigma}] &= i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho), \\
[P^\mu, P^\nu] &= 0.
\end{aligned}$$

Note. To characterize the Poincaré representation on fields we have considered the variation *at fixed coordinate* x, namely $\delta_0\phi(x)$, not the variation $\delta\phi(x) \equiv \phi'(x') - \phi(x) = 0$: this is because the transformation $x \longrightarrow x'$ simply corresponds to a change of reference frame, i.e. to expressing the position of a given point P, with coordinate x

according to observer O, in terms of the coordinates x' of observer O'. In doing so, the point P is kept fixed, so studying the variation $\delta\phi(x)$ corresponds to studying how a single degree of freedom (the value of the field at fixed point P) changes under change of parametrization. The basis for this representation is thus one dimensional, and since $\delta\phi(x) = 0$ the generators in this representation are zero: this is called scalar representation. Conversely, considering the variation keeping fixed the coordinate x, not the point P, means that we are comparing the field at different points, i.e. the point P which is called x by observer O, and the point P' which is called x by observer O'; in this case the base space is the set of functions $\phi(P)$, with P varying in space-time, thus this gives the infinite dimensional representation of the Poincaré group on fields, which is what we were looking for.

A recommended reading on the Lorentz representation on scalar fields is: *M. Maggiore, A Modern Introduction to Quantum Field Theory*, chapters 2.6.1 and 2.7.1.