Quantum Field Theory

Homework 3: solutions

Exercise 1

Given the Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$
$$\psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1}x') = e^{-\frac{1}{2}(i\theta^i+\eta^i)\sigma^i}\psi_L(\Lambda^{-1}x'),$$

where θ, η are the parameters associated respectively to rotations and boosts, one can consider the bilinear $\psi_L^\dagger\bar\sigma^\mu\psi_L$, where as usual $\bar\sigma^\mu = (1,\, -\sigma^i)$. We recall the notation for spinorial indices:

$$\psi_{L\,\alpha}, \qquad \psi^\dagger_{L\,\dot\beta}, \qquad (\bar\sigma^\mu)^{\dot\beta\alpha},$$

since the † transforms undotted indexes in dotted ones and vice versa. Thus the transformation properties of the bilinear are:

$$\psi'^\dagger_{L\,\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}\psi'_{L\,\alpha} = \psi^\dagger_{L\,\dot\gamma}(\Lambda^\dagger_L)^{\dot\gamma}_{\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}(\Lambda_L)^\delta_\alpha\psi_{L\,\delta}.$$

We now recall that $\Lambda^\dagger_L\bar\sigma^\mu\Lambda_L = \Lambda^\mu{}_\nu\bar\sigma^\nu$ to conclude that this bilinear transforms in the representation (1/2, 1/2). This is the expected results since all the spinorial indices are contracted while one vector index is free (in practice the $\bar\sigma^\mu$ represent the Clebsch-Gordan coefficient needed to pass from the $(1/2,\,0)\otimes(0,\,1/2)$ to the (1/2, 1/2)). Let us consider now the left doublet Ψ_L and the right singlet ψ_R with transformation properties under $SU(2)$ and Lorentz given by:

$$\text{Lorentz}\begin{cases} x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ (\Psi'_L)^a_\alpha(x') = (\Lambda_L)_\alpha^{\ \beta}(\Psi_L)^a_\beta(\Lambda^{-1}x') \\ (\psi'_R)^{\dot\beta}(x') = (\Lambda_R)^{\dot\beta}_{\ \dot\delta}(\psi_R)^{\dot\delta}(\Lambda^{-1}x') \end{cases}$$

$$\text{Isospin}\begin{cases} x'^\mu = x^\mu \\ (\Psi'_L)^a_\alpha(x') = U^a_{\ b}(\Psi_L)^b_\alpha(x) \\ (\psi'_R)^{\dot\beta}(x') = (\psi_R)^{\dot\beta}(x) \end{cases}$$

Hence the bilinear $\psi^\dagger_R\Psi_L = \psi^{\dagger\,\alpha}_R\Psi_{L\,\alpha}$ transforms as:

$$\text{Lorentz:}\qquad \psi'^{\dagger\,\alpha}_R\Psi'_{L\,\alpha} = (\Lambda_R\psi_R)^{\dagger\,\alpha}(\Lambda_L\Psi_L)_\alpha = \psi^{\dagger\gamma}_R(\Lambda^\dagger_R)_\gamma^{\ \alpha}(\Lambda_L)_\alpha^{\ \beta}\Psi_{L\,\beta} = \psi^{\dagger\,\gamma}_R\Psi_{L\,\gamma},$$

since $\Lambda^\dagger_R = \Lambda^{-1}_L$. Hence the latter is a scalar under Lorentz transformations. Note that we have omitted the x dependence but clearly the complete relation would be:

$$\psi'^\dagger_R(x')\Psi'_L(x') = \psi^\dagger_R(\Lambda^{-1}x')\Psi_L(\Lambda^{-1}x'),$$

which is the usual one for scalar quantities. Under $SU(2)$ transformation one gets:

$$\text{Isospin:}\qquad \psi'^\dagger_R\Psi'^a_L = \psi^\dagger_R U^a_{\ b}\Psi^b_L = U^a_{\ b}\psi^\dagger_R\Psi^b_L.$$

therefore the bilinear is an Isospin doublet. Still, this was expected since the spinor indices are all contracted while, concerning $SU(2)$, we are considering the product: $0\otimes 1/2 = 1/2$.

Let's consider finally the term $\Psi^\dagger_L\sigma^i\,\partial\!\!\!/\Psi_L$:

$$(\Psi^\dagger_L)_{\dot\alpha\,b}(\sigma^i)^b_{\ a}(\bar\sigma^\mu)^{\dot\alpha\beta}\partial_\mu(\Psi_L)^a_\beta,$$

where by convention the † exchanges dotted with undotted indices and lowers the Isospin index. Then the two transformations give:

$$\text{Lorentz:} \quad (\Psi'^\dagger_L)_{\dot\alpha\, b}(\sigma^i)^b{}_a(\bar\sigma^\mu)^{\dot\alpha\beta}\partial'_\mu(\Psi'_L)^a_\beta = (\Psi^\dagger_L)_{\dot\gamma}(\Lambda^\dagger_L)^{\dot\gamma}{}_{\dot\alpha}\sigma^i(\bar\sigma^\mu)^{\dot\alpha\beta}\Lambda_\mu{}^\nu\partial_\nu(\Lambda_L)_\beta{}^\delta(\Psi_L)_\delta = \Lambda^\mu{}_\nu\Lambda_\mu{}^\rho\Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\rho\Psi_L$$
$$= (\Lambda^{-1})_\nu{}^\mu\Lambda_\mu{}^\rho\Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\rho\Psi_L = \Psi^\dagger_L\sigma^i\bar\sigma^\nu\partial_\nu\Psi_L,$$
$$\text{Isospin:} \quad (\Psi'^\dagger_L)_b(\sigma^i)^b{}_a\,\partial\!\!\!/(\Psi'_L)^a = (\Psi^\dagger_L)_c(U^\dagger)^c{}_b(\sigma^i)^b{}_a\,\partial\!\!\!/ U^a{}_d(\Psi_L)^d = R^{(j=1)}[U]^i{}_j(\Psi^\dagger_L)_b(\sigma^j)^b{}_a\,\partial\!\!\!/(\Psi_L)^a,$$

where we have made use of the relations:

$$(\Lambda^\dagger_L)^{\dot\gamma}{}_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}(\Lambda_L)_\beta{}^\delta = \Lambda^\mu{}_\nu(\bar\sigma^\mu)^{\dot\gamma\delta},$$
$$(U^\dagger)^c{}_b(\sigma^i)^b{}_aU^a{}_d = R^{(j=1)}[U]^i{}_j(\sigma^j)^c{}_d.$$

In the end the latter bilinear is a Lorentz scalar while is an Isospin vector (that is to say it transforms in the $j=1$ representation of $SU(2)$).

Exercise 2

A generic tensor transforming in an irreducible representation $(j/2, \bar{j}/2)$ of the Lorentz group (or, more precisely, of $SL(2,\mathbb{C})$), is an object with j and $\bar{j}$ unprimed and primed indices which transforms as follows:

$$t_{A_1\ldots A_j}^{A'_1\ldots A'_{\bar j}} \longrightarrow (\Lambda_L)_{A_1}^{\ B_1}\ldots(\Lambda_L)_{A_j}^{\ B_j}(\Lambda_R)^{A'_1}_{\ B'_1}\ldots(\Lambda_R)^{A'_{\bar j}}_{\ B'_{\bar j}}\, t_{B_1\ldots B_j}^{B'_1\ldots B'_{\bar j}}.$$

Notice that for this tensor to belong to the specified irreducible representations all indices must be symmetrized. Indeed any pair anitisymmetric of indices must be proportional to the tensors ϵ_{AB} or $\epsilon^{A'B'}$, which are invariant under $SL(2,\mathbb{C})$ transformations.
A generic tensor with latin indices can be decomposed into irreducible representations explicitly via contraction with the Pauli Matrices $\sigma^\mu_{AA'}$ and $\bar\sigma^{\mu\ A'A}$. For instance a vector V_μ belongs to a $(1/2,1/2)$ irreducible representation, since it can be written as

$$V_{AA'} \equiv \sigma^\mu_{AA'}V_\mu$$

which transform as (recall that indices can be raised and lowered using ϵ_{AB}, ϵ^{AB}, $\epsilon_{A'B'}$, $\epsilon^{A'B'}$)

$$V_{AA'} \longrightarrow (\Lambda_L)_A^{\ B}(\Lambda_R)_{A'}^{\ B'}V_{BB'}.$$

Finally, since the algebra of $SL(2,\mathbb{C})$ is isomorphic to the complexified sum of two independent $SU(2)$ subalgebras,

$$\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2) \oplus_{\mathbb{C}} \mathfrak{su}(2),$$

tensor pruducts can be formally decomposed using the usual $SU(2)$ rule both for j and $\bar{j}$:

$$\begin{aligned}(j_1,\bar j_1)\otimes(j_2,\bar j_2) &= (j_1\otimes j_2, \bar j_1\otimes\bar j_2)\\ &= (|j_1-j_2|\oplus|j_1-j_2|+1\oplus\ldots\oplus j_1+j_2, |\bar j_1-\bar j_2|\oplus|\bar j_1-\bar j_2|+1\oplus\ldots\oplus\bar j_1+\bar j_2)\\ &= (|j_1-j_2|,|\bar j_1-\bar j_2|)\oplus(|j_1-j_2|+1,|\bar j_1-\bar j_2|)\oplus\ldots\oplus(j_1+j_2,\bar j_1+\bar j_2).\end{aligned}$$

Here we formally wrote, abusing of notation, $(j_a\oplus j_b, \bar j_a\oplus\bar j_b)$ to really mean $(j_a,\bar j_a)\oplus(j_b,\bar j_a)\oplus(j_a,\bar j_b)\oplus(j_b,\bar j_b)$.

Let us now discuss two applications.

- Consider the product of a vector and a left handed spinor:

  $$A_\mu\psi_A.$$

  The tensor product decomposition can be computed formally as:

  $$(1/2,1/2)\otimes(1/2,0) = (1/2\otimes1/2, 1/2) = (0\oplus1, 1/2) = (0,1/2)\oplus(1,1/2).$$

The two representations can be obtained explicitly considering

$$\sigma^\mu_{BB'} V_\mu \psi_A = V_{BB'}\psi_A \equiv \Phi_{AB\ B'}.$$

Symmetrizing and antisymmetrizing indices the representations are obtained:

$$\Phi_{(AB)\ B'} \equiv \frac{1}{2}\left(\Phi_{AB\ B'} + \Phi_{BA\ B'}\right) \ \sim\ (1,1/2),$$

$$\Phi_{[AB]\ B'} \equiv \frac{1}{2}\left(\Phi_{AB\ B'} - \Phi_{BA\ B'}\right) = \frac{1}{2}\epsilon_{AB}\ \Phi_{B'} \ \sim\ (0,1/2).$$

where we defined $\Phi_{B'} \equiv \epsilon^{AB}\Phi_{AB\ B'}$ and we used the fact that any $2d$ antisymmetric tensor must be proportional to ϵ_{AB}; the $1/2$ in the last equation is fixed contracting with ϵ^{AB}. Finally we use parenthesis ( ) for symmetrization of indices, square parenthesis [ ] for antisymmetrization.

- Consider now the product

$$A_\mu B_\nu.$$

Recalling that a vector belongs to $(1/2,1/2)$, the tensor product gives:

$$(1/2,1/2)\otimes(1/2,1/2) = (0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1).$$

To write explicitly the tensor product decomposition one could proceed systematically, but there is a simpler (and more instructive) way. Recall that Lorentz transformations preserve the symmetricity or antisymmetricity of indices. Also, the trace of a tensor is trivially left unchanged by Lorentz transformations. Hence we can divide the product $A_\mu B_\nu$ into three tensors which transform independently:

$$\begin{aligned} S_{\mu\nu} &\equiv A_{(\mu}B_{\nu)} - \frac{1}{4}\eta_{\mu\nu} A\cdot B \\ A_{\mu\nu} &\equiv A_{[\mu}B_{\nu]} \\ T &\equiv \eta^{\mu\nu}A_\mu B_\nu = A\cdot B. \end{aligned}$$

These are, respectively, the traceless symmetric part, the antisymmetric part and the trace. Now we can match with the decomposition we found before simply counting the components of these tensors. Indeed we know from the theory of $SU(2)$ that a $(0,0)$ representation (scalar) has just one component, the representations $(1,0)$ and $(0,1)$ have 3 each, while the representation $(1,1)$ has $3\times 3 = 9$ components. Then, after counting the components of the tensors, the only possible identifications are:

$$S_{\mu\nu}\ \sim\ (1,1), \qquad A_{\mu\nu}\ \sim\ (1,0)\oplus(0,1), \qquad T\ \sim\ (0,0).$$

This means that $S_{\mu\nu}$ can be written as a tensor $S^{A'B'}_{AB}$, $A_{\mu\nu}$ as $A_{AB}\epsilon^{A'B'} + A^{A'B'}\epsilon_{AB}$ and T is a scalar. To be more explicit, we can define $S^{A'B'}_{AB}$ as:

$$\sigma^\mu_{AA'}\sigma^\nu_{BB'}S_{\mu\nu} \equiv S_{AB\ A'B'}\ \sim\ (1,1).$$

To write the antisymmetric part, we notice that the only two objects at our disposal to convert an antisymmetric tensor in latin indices to $(0,1)$ and $(1,0)$ representations are

$$\left(\sigma^{\mu\nu}\right)_A^{\ B} = \frac{1}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_A^{\ B}, \qquad \left(\bar\sigma^{\mu\nu}\right)^{A'}_{\ B'} = \frac{1}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)^{A'}_{\ B'}.$$

Then we can define

$$\left(\sigma^{\mu\nu}\right)_A^{\ B}A_{\mu\nu} \equiv A_A^{\ B}\ \sim (1,0), \qquad \left(\bar\sigma^{\mu\nu}\right)^{A'}_{\ B'}A_{\mu\nu} \equiv A^{A'}_{\ B'}\ \sim (0,1).$$

Exercise 3

- Under a generic Lorentz transformation Λ^μ_ν the two tensors under consideration simply transform as follows

$$p^\mu \to \Lambda^\mu_{\ \nu}\, p^\nu, \qquad j^{\mu\nu} \to \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\, j^{\alpha\beta}. \tag{1}$$

- As we discussed in previous exercises we can construct the two complex combinations $j^{\pm}$ that, under a Lorentz transformation, simply rotate according to two independent $SU(2)$ groups (remember the isomorphism $SU(2)_L \times SU(2)_R \simeq SO(1,3)$). In particular we know that they transform as vector in the adjoint representations of these $SU(2)_L \times SU(2)_R$, i.e. usual $SO(3)$ rotations. Therefore it is easy to see that we can independently rotate the two $j^{\pm}$ in the 1 direction, to be in the form

$$j^+ = \begin{pmatrix} j_1^+ \\ 0 \\ 0 \end{pmatrix}, \qquad j^- = \begin{pmatrix} j_1^- \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

In this frame we have $k_2 = k_3 = j_2 = j_3 = 0$ and then j^μ takes the form

$$j^{\mu\nu} = \begin{pmatrix} 0 & -k_1 & 0 & 0 \\ k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -j_1 \\ 0 & 0 & j_1 & 0 \end{pmatrix}. \tag{3}$$

Notice that this tensor cannot be further reduced by doing more Lorentz transformations.

- The residual symmetry group is the subgroup of Lorentz which leaves invariant this form of $j^{\mu\nu}$. This corresponds to the transformation that leave invariant the form of $j^{\pm}$ of eq (2), that is the rotations around the two 1 directions in the $SU(2)_L \times SU(2)_R$ language. As usual Lorentz transformations these corresponds to boosts and rotations along the first direction.

- We can use this residual symmetry to reduce p^μ while keeping $j^{\mu\nu}$ in the reduced form. First we can use the rotation around 1 to align the (p_2, p_3) components along the 3 direction. In other words we can go to a frame where $p_2 = 0$. We can then similarly apply a boost along 1 to go to a frame where also $p_1 = 0$. In this frame we have the fully reduced tensors

$$j^{\mu\nu} = \begin{pmatrix} 0 & -k_1 & 0 & 0 \\ k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -j_1 \\ 0 & 0 & j_1 & 0 \end{pmatrix} \qquad p^\mu = \begin{pmatrix} p_0 \\ 0 \\ 0 \\ p_3 \end{pmatrix}. \tag{4}$$

- In this special frame, where the only not null components of j and p are p_0, p_3, j_1 and k_1, the four invariants given in the exercise take the form

$$e_1 = j_{\mu\nu} j^{\mu\nu} = 2 j_{23} j^{23} + 2 j_{10} j^{10} = 2 j_1^2 - 2 k_1^2 \,, \tag{5}$$

$$e_2 = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} j_{\mu\nu} j_{\rho\sigma} = 2 j_{01} j_{23} = -2 k_1 j_1 \,, \tag{6}$$

$$e_3 = p^\mu p_\mu = (p^0)^2 - (p^3)^2 \,, \tag{7}$$

$$e_4 = w^\mu w_\mu = -(w^1)^2 - (w^2)^2 = -(p^0)^2 (j_1)^2 - (k_1)^2 (p^3)^2 \,. \tag{8}$$

In the last step, we made use that the only non-zero compontens of w^μ are $w^1 = \epsilon^{1230} j_{23} p_0 = p_0 j_1$ and $w^2 = -\epsilon^{0123} j_{10} p_3 = -k_1 p^3$, in this frame.

- To check that the map between the four invariants and the four components in the special frmae is invertible we check if the Jacobian determinant of this map $\det J$ is non-zero. The computation is straightforward and gives

$$\det J = 32 p^0 p^3 (j_1^2 + k_1^2) \neq 0 \,. \tag{9}$$

- We now have to find which of $E_1, \ldots E_4$ are Casimirs of the Poincaré group. E_i is a Casimir if

$$[E_i, P^\mu] = [E_i, J^{\mu\nu}] = 0 \,. \tag{10}$$

Since each E_i is built out of P^μ and $J^{\mu\nu}$ we can directly use the Poincaré algebra to check the equation above.

Note, however, that each E_i is a Lorentz scalar, meaning that, by construction, they are invariant under Lorentz transformations. Therefore $[E_i, J^{\mu\nu}] = 0$. The commutator with P^μ, instead, requires explicit computation.

It is trivial to see that $[E_3, P^\mu] = 0$ and, as we have shown in exercise 1 of set 12, also $[E_4, P^\mu] = 0$.

Using the Poincaré algebra we however find that

$$[E_1, P_\mu] = 2i(J_{\alpha\mu}P^\alpha + P^\alpha J_{\alpha\mu}) \neq 0, \qquad [E_2, P_\mu] = -4iW_\mu \neq 0\,. \tag{11}$$

- Up until now we have shown that, given the two tensors P^μ and $J^{\mu\nu}$, we can build only four independent Lorentz invariant, meaning that any other invariant can be always written as a function of these four objects. This means that there are at most 4 Casimirs, and we have explicitely found 2 of them. The last step is to show that the Casmirs are two and only two. Since there are only four independent invariants, if there exist a third Casimir then it will be function $f(E_{i=1,\ldots,4})$. Therefore we have to show that no f function of E_1 and E_2 commutes with P^μ.

  Without loss of generality we can consider a function f of the form

$$f(E_1, E_2) = g_e(E_1) + g_o(E_1)E_2\,, \tag{12}$$

  since any power of $E_2^n = \left(J^{\mu\nu}\tilde{J}_{\mu\nu}\right)^n$ $(n > 1)$ can be rewritten as E_2 times some polynomial of E_1, using the schematic identity $\epsilon\epsilon \sim \delta\delta\delta\delta + perm..$, with ϵ the Levi-Civita tensor and δ the Kronecker delta. We can take

$$g_e(E_1) = \sum_{n=1}^{\infty} c_e^{(n)} E_1^n\,, \qquad\qquad g_o(E_1) = \sum_{n=0}^{\infty} c_o^{(n)} E_1^n\,, \tag{13}$$

  where $c_e^{(n)}$ and $c_o^{(n)}$ are themselves arbitrary functions of E_3 and E_4. Since they both commute with P^μ they can be treated as numerical coefficients.

  We have

$$[f(E_1, E_2), P_\mu] = [g_e(E_1), P_\mu] + [g_o(E_1), P_\mu]\, E_2 + g_o(E_1)\,[E_2, P_\mu]\ . \tag{14}$$

  Let us start from the first term, according to eq. (11) we have that

$$[E_1, P_\mu] = (-4iJ_{\mu\alpha} + 6\eta_{\mu\alpha})\, P^\alpha \equiv A_{\alpha\beta}P^\alpha\,, \tag{15}$$

  where obviously $[A_{\alpha\beta}, E_1] = 0$, since E_1 is a Lorentz invariant. Moreover

$$\begin{aligned}\left[E_1^2, P_\alpha\right] &= E_1\,[E_1, P_\mu] + [E_1, P_\mu]\, E_1 = E_1 A_{\alpha\mu}P^\alpha + A_{\alpha\mu}P^\alpha E_1\\ &= E_1 A_{\alpha\mu}P^\alpha - A_{\alpha\mu}\,[E_1, P^\alpha] + A_{\alpha\mu}E_1P^\alpha = 2E_1A_{\alpha\mu}P^\alpha - A_{\beta\mu}A^{\alpha\beta}P_\alpha\,,\end{aligned} \tag{16}$$

  and in the same way you can compute

$$[E_1^n, P_\alpha] = \# E_1^{n-1} A_{\mu\alpha}P^\mu + \ldots\,, \tag{17}$$

  where the # is simply a number. Therefore, if we assume $g_e(E_1)$ to be a polynomial of degree n in E_1, the we will find

$$[g_e(E_1), P_\alpha] = \# E_1^{n-1} A_{\mu\alpha}P^\mu + \ldots\,, \tag{18}$$

  where the ... contains terms with less powers of J (notice that $A_{\mu\nu}$ is linear in J). The same argument can be applied to the second term in eq. (14). The third term is instead of the form $g_0(E_1)W_\mu$ from eq. (11).

  We have thus found that the commutator (14) is a sum of a generic non-zero function of E_1, of a second non-zero function of E_1 times E_2 and another piece proportional to W^μ. In this form it is clear that the only way for this commutator to be always zero is if both g_e and g_o are identically zero. We can then conclude that we cannot find a third independent Casimir of the Poincaré group.