Quantum Field Theory

Homework 2: solutions

Exercise 1

The action for a free massless scalar field in d dimension is

$$\mathcal{S} = \frac{1}{2}\int dt d^{d-1}x \; \partial_\mu\phi(x)\partial^\mu\phi(x).$$

We consider the scale transformation labelled by the parameter $\lambda \in \mathbb{R}$ and defined as

$$\begin{aligned} x'^\mu &= e^\lambda x^\mu, \\ \phi'(x') &= e^{k\lambda}\phi(x) = e^{k\lambda}\phi(e^{-\lambda}x'). \end{aligned}$$

Expanding for infinitesimal parameter one gets

$$\begin{aligned} x'^\mu &\simeq x^\mu + \lambda x^\mu + O(\lambda^2) \implies \epsilon^\mu = -x^\mu, \\ \phi'(x) &= (1+k\lambda+O(\lambda^2))\phi(x-\lambda x+..) \simeq (1+k\lambda+O(\lambda^2))(\phi(x) - \lambda x^\mu\partial_\mu\phi(x) + O(\lambda^2)) \\ &\simeq \phi(x) + k\lambda\phi(x) - \lambda x^\mu\partial_\mu\phi(x) + O(\lambda^2) \implies \Delta(x) = k\phi(x) - x^\mu\partial_\mu\phi(x). \end{aligned}$$

Here the usual indices a, i labeling different fields and parameters have disappeared since they assume only the value $a = i = 1$. In order to define a symmetry of the theory these transformation must leave invariant the action:

$$\begin{aligned} & x' = e^\lambda x \qquad d^dx' = e^{d\lambda}d^dx \qquad \partial'_\mu = e^{-\lambda}\partial_\mu, \\ & \partial'_\mu\phi'(x') = e^{k\lambda}\partial'_\mu\phi(x) = e^{(k-1)\lambda}\partial_\mu\phi(x), \\ & \frac{1}{2}\int d^dx\partial_\mu\phi(x)\partial^\mu\phi(x) \longrightarrow \frac{1}{2}\int d^dx\partial_\mu\phi(x)\partial^\mu\phi(x)e^{(2k-2+d)\lambda} \\ & \implies (2k-2+d)\lambda = 0 \implies k = 1 - \frac{d}{2}. \end{aligned}$$

In last equation we have discarded the solution $\lambda = 0$, which corresponds to the identical transformation, which is always an uninteresting symmetry.
In four dimension, $k = -1$ and the Noether's current reads

$$\begin{aligned} S^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta - \epsilon^\mu\mathcal{L} = -\left(\phi + x^\nu\partial_\nu\phi\right)\partial^\mu\phi + \frac{1}{2}x^\mu\left(\partial_\nu\phi(x)\partial^\nu\phi(x)\right) \\ &= -\phi\partial^\mu\phi - x^\nu\partial_\nu\phi\partial^\mu\phi + \frac{1}{2}x^\mu\partial_\nu\phi\partial^\nu\phi. \end{aligned}$$

Recalling the definition of the energy momentum tensor associated to this Lagrangian

$$T^\mu{}_\rho = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L} = \partial_\rho\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\rho\left(\partial_\nu\phi\partial^\nu\phi\right),$$

one has

$$T^\mu{}_\mu = \partial_\mu\phi\partial^\mu\phi - \frac{4}{2}\partial_\nu\phi\partial^\nu\phi = -\partial_\nu\phi\partial^\nu\phi\ .$$

One can consider an *improved energy momentum tensor* $K^\mu{}_\rho$ adding the terms

$$K^\mu{}_\rho = T^\mu{}_\rho + A\delta^\mu_\rho\Box\phi^2 + B\partial_\rho\partial^\mu\phi^2.$$

The choice of the constant A, B is fixed by the requirement that the above expression be conserved (as the original energy-momentum tensor) and in addition traceless:

$$\begin{aligned} & \partial_\mu K^\mu{}_\rho = \partial_\mu T^\mu{}_\rho + (A+B)\partial_\rho\Box\phi^2 = 0 \implies A + B = 0, \\ & K^\mu{}_\mu = T^\mu{}_\mu + 4A\Box\phi^2 - A\Box\phi^2 = 0. \end{aligned}$$

Using the identity
$$\Box\phi^2 = 2\partial_\mu\phi\partial^\mu\phi + 2\phi\Box\phi,$$
and making use of the equation of motion $\Box\phi = 0$, we can write the trace of the improved energy momentum tensor as

$$K^\mu{}_\mu = T^\mu{}_\mu + 6A\partial_\mu\phi\partial^\mu\phi = (-1+6A)\partial_\mu\phi\partial^\mu\phi = 0 \implies A = \frac{1}{6}.$$

At the end the improved energy momentum tensor reads

$$K^\mu{}_\rho = \partial_\rho\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\rho\left(\partial_\nu\phi\partial^\nu\phi\right) + \frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right).$$

We can write the dilatations current S^μ in terms of the above improved energy momentum tensor

$$S^\mu = -x^\nu K^\mu_\nu - \phi\partial^\mu\phi + x^\rho\frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right).$$

The invariance of the theory under scale transformations implies the vanishing of $\partial_\mu S^\mu$ and therefore

$$\begin{aligned} 0 = \partial_\mu S^\mu &= -K^\mu{}_\mu - x^\nu\partial_\mu K^\mu{}_\nu - \partial_\mu\phi\partial^\mu\phi + \frac{1}{6}(4\Box\phi^2 - \Box\phi^2) \\ \implies K^\mu{}_\mu &= 0 \end{aligned}$$

where we have again expanded $\Box\phi^2$ and used the equation of motion $\Box\phi = 0$ and the conservation of K^ν_μ. The invariance of the theory under dilatations forces the improved energy momentum tensor to be traceless. For free theories we already know that this is the case since $K^\mu{}_\nu$ has been constructed in such a way as to have this property. However one could extend the definition of K for a more general theory with a potential

$$K^\mu{}_\rho = \partial_\rho\phi\partial^\mu\phi - \delta^\mu_\rho\left(\frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - V\right) + \frac{1}{6}\left(\delta^\mu_\rho\Box\phi^2 - \partial_\rho\partial^\mu\phi^2\right),$$

and it is possible to check that the tracelessness of K^μ_ν represents a non trivial constraint on the potential V.

The addition of a potential of the form $c_n\phi^n$ brings an additional constraint between k and d which can fix definitively the dimension. In order to have an invariant theory one needs:

$$\int d^dx'\phi'^n(x') = e^{d\lambda+nk\lambda}\int d^dx\phi^n(x) = \int d^dx\phi^n(x) \implies \begin{cases} d+nk=0 \\ k = 1-\frac{d}{2}. \end{cases}$$

The solution for the above system of equation doesn't exist for $n=2$. Instead:

$$\begin{aligned} &\text{For } n=3 \;\Rightarrow\; d=6, \\ &\text{For } n=4 \;\Rightarrow\; d=4. \end{aligned}$$

The dimensions in energy of the parameters appearing in the potential are then:

$$\begin{aligned} &[\text{Action}] = E^0, \qquad [d^dx] = E^{-d}, \qquad [\mathcal{L}] = E^d, \\ &[\partial] = E, \qquad [\phi] = E^{\frac{d}{2}-1}, \\ &[m] = E, \qquad [\beta] = E^{3-\frac{d}{2}}, \qquad [\alpha] = E^{4-d}. \end{aligned}$$

Therefore the couplings α, β are both adimensional in the dimension in which the Lagrangian is invariant under scale transformation. This is not unexpected because the scale transformation deforms lengths and energies as well. The invariance of the theory under such transformation means that the dynamics is the same at all energy scales. In order for this to be true there mustn't be any reference scale in the theory. Therefore in a scale invariant theory only dimensionless parameters are allowed in the potential. This also explains why there is no solution for the term $m^2\phi^2$: the dimension of m doesn't depend on the dimension d, hence it always introduces a reference scale which is the indeed the mass of the field.

Exercise 2

Consider a symmetry defined by the transformation acting on fields:

$$\begin{aligned} x' &= x, \\ \phi'_a(x') &= \mathcal{R}_a^{\ b}\phi_b(x) \simeq \phi_a(x) + i\alpha^A (T^A)_a^{\ b}\phi_b(x), \end{aligned}$$

where $(T^A)_a^{\ b}$ are the generators of the symmetry in the appropriate representation and satisfy the Lie algebra with the ordinary commutator: $[T^A, T^B] = if^{ABC}T^C$. One can easily compute the conserved Noether's charge:

$$Q^A = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \Delta_a(x) \right) = i \int d^3x \, \pi^a (T^A)_a^{\ b}\phi_b(x).$$

Therefore the Poisson brackets between two charges give:

$$\{Q^A, Q^B\} = \int d^3z \left(\frac{\delta Q^A}{\delta \pi^c(z)} \frac{\delta Q^B}{\delta \phi_c(z)} - \frac{\delta Q^A}{\delta \phi_c(z)} \frac{\delta Q^B}{\delta \pi^c(z)} \right).$$

Since

$$\begin{aligned} \frac{\delta Q^A}{\delta \pi(z)^c} &= i\frac{\partial \left(\pi^a (T^A)_a^{\ b}\phi_b\right)}{\partial \pi^c}(z) = i(T^A)_c^{\ b}\phi_b(z), \\ \frac{\delta Q^B}{\delta \phi(z)_c} &= i\frac{\partial \left(\pi^a (T^B)_a^{\ b}\phi_b\right)}{\partial \phi_c}(z) = i\pi^a(z)(T^B)_a^{\ c}, \end{aligned}$$

hence:

$$\{Q^A, Q^B\} = \int d^3z \; \pi^a \left[T^A, T^B\right]_a^{\ b} \phi_b = if^{ABC} \int d^3z \pi^a (T^C)_a^{\ b}\phi_b = f^{ABC}Q^C.$$

One can finally define $Q^A = -i\tilde{Q}^A$ so that

$$\{\tilde{Q}^A, \tilde{Q}^B\} = if^{ABC}\tilde{Q}^C.$$

There is however a shorter way to obtain the commutation rules for the charges and it involves the Jacobi identity; recall indeed that the Poisson brackets, as all the Lie products, satisfy the Jacobi relation:

$$\{\{Q^A, Q^B\}, \phi_a\} + \{\{Q^B, \phi_a\}, Q^A\} + \{\{\phi_a, Q^A\}, Q^B\} = 0.$$

Since the charges are the generators of the transformation:

$$\{Q^A, \phi_a\} = \Delta_a^A = i(T^A)_a^{\ b}\phi_b,$$

then, applying two times this definition one gets

$$\{\{Q^A, Q^B\}, \phi_a\} = -(T^B)_a^{\ c}(T^A)_c^{\ b}\phi_b + (T^A)_a^{\ c}(T^B)_c^{\ b}\phi_b = if^{ABC}(T^C)_a^{\ b}\phi_b = f^{ABC}\{Q^C, \phi_a\} = \{f^{ABC}Q^C, \phi_a\}.$$

Exercise 3

Let us recall that the group $SO(N)$ is defined as:

$$SO(N) = \{O : \; OO^T = \mathbb{1}, \; \det(O) = 1\}.$$

The case $N = 1$ corresponds to the trivial group and thus we get the most general Lagrangian with terms whose dimension is less or equal than four:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4.$$

We did not write a linear term $\mu^3\phi$ since this can always be eliminated shifting $F\phi \to \phi - \mu^3/m^2$. Also, we did not write total derivatives like $n\partial^2\phi$.
For $N \geq 2$, we can build invariants contracting the two invariant tensors of $O(N)$:

$$\delta_i^j, \quad \epsilon^{i_1 \dots i_N}.$$

Contracting the first we get the invariants, with $d \leq 4$:

$$\partial_\mu \Phi^T \partial^\mu \Phi, \quad \Phi^T \Phi, \quad (\Phi^T \Phi)^2.$$

The epsilon tensor instead does not give non vanishing invariants. For instance $\epsilon^{i_1 \cdots i_N} \phi_{i_1} \dots \phi_{i_N} = 0$ by antisymmetry. Then we can write The $SO(N)$ model Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{m^2}{2}\Phi^T \Phi - \frac{\lambda}{4}(\Phi^T \Phi)^2. \tag{1}$$

This Lagrangian is really invariant under $O \in O(N)$, i.e. also under transformations such that $\det(O) = -1$. Indeed the only requirement for $\Phi^T\Phi$ to be invariant is $O^T O = \mathbb{1}$.

Now we want to build the most general Lorentz invariant Lagrangian of two scalars with terms up to dimension 4, that is symmetric under the following three transformations separately:

1. $\phi_1 \to -\phi_1$
2. $\phi_2 \to -\phi_2$
3. $\phi_1 \leftrightarrow \phi_2$

The first two transformations imply that we can only write terms which are separately quadratic in the fields. Taking into account the last one, we conclude that the required Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \lambda_1\left(\phi_1^4 + \phi_2^4\right) - \lambda_2 \phi_1^2 \phi_2^2. \tag{2}$$

Each of the three transformations above taken alone forms a group which is isomorphic to $\mathbb{Z}_2$. However combined together they form a group which is different from $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, since they do not commute with each other. Consider for instance

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \overset{3}{\to} \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix} \overset{2}{\to} \begin{pmatrix} \phi_2 \\ -\phi_1 \end{pmatrix}, \qquad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \overset{2}{\to} \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix} \overset{3}{\to} \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix}.$$

This is called the *dihedral* group $\mathbb{D}_4$ and describes the symmetry of a square.
Let us call D_1, D_2, D_3 the action of the three transformations on the fields

$$D_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_1 \\ \phi_2 \end{pmatrix}, \qquad D_2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix}, \qquad D_3 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}.$$

By combining the action of these, it is easy to see that the group is formed by eight elements. Indeed the most general transformation of the field doublet takes the form

$$D \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \pm\phi_{1/2} \\ \pm\phi_{2/1} \end{pmatrix}, \qquad D \in \mathbb{D}_4.$$

We can write all the elements as

$$\mathbb{D}_4 = \{\mathbb{1}, D_1, D_2, D_3, D_1 D_2, D_1 D_3, D_2 D_3, D_1 D_2 D_3\}.$$

It is now easy to check that by taking different products one does not get new elements. For instance the following hold

$$D_1 D_2 = D_2 D_1, \quad D_3 D_1 = D_2 D_3, \quad D_3 D_2 = D_1 D_3,$$
$$D_3 D_1 D_2 = D_3 D_2 D_1 = D_1 D_2 D_3, \qquad D_1 D_3 D_2 = D_2 D_3 D_1 = D_3.$$

We can build a matrix representation of this group, by looking at its action on the field doublet $(\phi_1, \phi_2)^T$:

$$D_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the most general element belonging to $\mathbb{D}_4$ takes the form

$$D = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \qquad \text{or} \qquad D = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Finally the Lagrangian (2) reduces to (1) when $\lambda_2 = 2\lambda_1$, in which case the symmetry group is enhanced to $O(2) \supset \mathbb{D}_4$.