Quantum Field Theory

Set 9: solutions

Exercise 1

Let's first introduce some notation for delta functions, used also in next exercise.

$$\int d^3x\; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}),$$
$$\int d^3k\; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{x}),$$
$$\int d^3x\; \delta^3(\vec{x}) = \int d^3k\; \delta^3(\vec{k}) = 1.$$

When $\vec{k}$ can assume only discrete values $\vec{k}_n = 2\pi\vec{n}/L$, the Dirac delta becomes a Kronecker delta, since it has to give 1 when summed, *not* when integrated, and the integral on momenta becomes a sum over numbers. Basically the discrete case can be deduced from the continuous one making the following formal replacements.

$$\delta^3(\vec{k}) \;\longrightarrow\; \left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}},$$
$$\int d^3k \;\longrightarrow\; \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{n}\in\mathbb{Z}^3},$$

and in particular

$$\int d^3x\; e^{i\vec{k}\cdot\vec{x}} \;\longrightarrow\; \int d^3x\; e^{i\vec{k}_n\cdot\vec{x}} = (2\pi)^3\left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}}.$$

Now one can compute explicitly the required expression:

$$\int d^3x \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{L^{3/2}}\phi_n(t)\partial_i\, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}} \sum_{\vec{m}\in\mathbb{Z}^3} \frac{1}{L^{3/2}}\phi_m(t)\partial_i\, e^{i\frac{2\pi}{L}\vec{m}\cdot\vec{x}} = \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m}\;\phi_n(t)\phi_m(t)\int d^3x\, e^{i\frac{2\pi}{L}(\vec{m}+\vec{n})\cdot\vec{x}}$$
$$= \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m}\;\phi_n(t)\phi_m(t)L^3\delta^3_{\vec{n}+\vec{m},\vec{0}} = \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{n}\in\mathbb{Z}^3} |\vec{n}|^2|\phi_n(t)|^2.$$

where we used the fact that $\phi_{-n}(t) = \phi_n^*(t)$ since $\phi(\vec{x},t)$ is real.

Exercise 2

Since ϕ is a real field we have $\phi(x,t) = \phi(x,t)^*$. Using this equality in the inverse Fourier transform

$$\phi(k,t) = \int d^3x e^{-ikx}\phi(x,t)$$

we find

$$\phi(k,t)^* = \int d^3x e^{ikx}\phi(x,t)^* = \int d^3x e^{-i(-k)x}\phi(x,t) = \phi(-k,t).$$

The same reasoning can be used for $\pi(k,t)$. Now we plug the Fourier transform of $\phi(x,t)$ in the formula for the Hamiltonian. Consider the first term

$$\int d^3x \pi(x)^2 = \int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3} e^{ix(k+q)}\pi(k)\pi(q).$$

We can now integrate in x to obtain a delta function that allows us to also integrate in q

$$\int d^3x \pi(x)^2 = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} (2\pi)^3 \delta(k+q) \pi(k)\pi(q) = \int \frac{d^3k}{(2\pi)^3} \pi(k)\pi(-k) = \int \frac{d^3k}{(2\pi)^3} |\pi(k)|^2,$$

where in the last equality we have used the identity just proved in the previous point. The same calculation can be used for the other two terms. The only small difference comes from the term with the gradient for which we have

$$\int d^3x (\nabla\phi(x))^2 = \int d^3x \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} e^{ix(k+q)} (-(k\cdot q)) \phi(k)\phi(q) = \int \frac{d^3k}{(2\pi)^3} k^2 |\phi(k)|^2.$$

In the end we find

$$H = \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} |\pi(k,t)|^2 + \frac{1}{2}(k^2+m^2)|\phi(k,t)|^2 \right].$$

Exercise 3

The first commutator is

$$\begin{aligned} [H, a(\vec{p})] &= \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left[a^\dagger(\vec{k}) a(\vec{k}), a(\vec{p}) \right] = \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left(a^\dagger(\vec{k}) \left[a(\vec{k}), a(\vec{p}) \right] + \left[a^\dagger(\vec{k}), a(\vec{p}) \right] a(\vec{k}) \right) \\ &= \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left(0 - (2\pi)^3 \delta^3(\vec{p}-\vec{k}) a(\vec{k}) \right) = -\omega(\vec{p}) a(\vec{p}). \end{aligned}$$

Analogously,

$$\begin{aligned} \left[H, a^\dagger(\vec{p})\right] &= \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left[a^\dagger(\vec{k}) a(\vec{k}), a^\dagger(\vec{p}) \right] = \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left(a^\dagger(\vec{k}) \left[a(\vec{k}), a^\dagger(\vec{p}) \right] + \left[a^\dagger(\vec{k}), a^\dagger(\vec{p}) \right] a(\vec{k}) \right) \\ &= \int \frac{d^3k}{(2\pi)^3}\, \omega(\vec{k}) \left(a^\dagger(\vec{k}) (2\pi)^3 \delta^3(\vec{p}-\vec{k}) + 0 \right) = +\omega(\vec{p}) a^\dagger(\vec{p}). \end{aligned}$$

Exercise 4

We want to show the invariance under Lorentz transformations of the measure over momentum space $\frac{d^3k}{(2\pi)^3 2k_0}$, where $k_0 \equiv \omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$ is the energy associated to a given particle of mass m. There are two ways to achieve the result: the first consists in checking explicitly the invariance performing a Lorentz transformation on momenta; however we first prove it performing a manipulation. The measure can be rewritten as

$$\frac{d^3k}{(2\pi)^3 2k_0} = \frac{d^3k}{(2\pi)^3}\, dk_0\, \delta(k^2 - m^2)\theta(k_0) = \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2)\theta(k_0). \tag{1}$$

In order to convince oneself that this is true, one can consider a test function f of momenta and integrate it over k_0:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \delta(k_0^2 - |\vec{k}|^2 - m^2)\theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3} dk_0 \left(\frac{\delta(k_0 + \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} + \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \right) \theta(k_0) f(\vec{k}, k_0),$$

where we have used the well known relation for the δ function: given a function $g(x)$ which vanishes in the points $\{x_1, ..., x_n\}$, then

$$\delta(g(x)) = \sum_{i=1}^{n} \delta(x - x_i) \frac{1}{|g'(x_i)|}.$$

In the present case the equation $(k_0)^2 - |\vec{k}|^2 - m^2 = 0$ admits two opposite solutions, hence the two terms in the parenthesis, but the theta function gets rid of the second one since in that case $k_0 < 0$. Finally, integrating on k_0 one gets the initial measure:

$$\int \frac{d^3k}{(2\pi)^3} dk_0 \frac{\delta(k_0 - \sqrt{|\vec{k}|^2 + m^2})}{2|k_0|} \theta(k_0) f(\vec{k}, k_0) = \int \frac{d^3k}{(2\pi)^3 2\sqrt{|\vec{k}|^2 + m^2}} f(\vec{k}, \sqrt{|\vec{k}|^2 + m^2}).$$

Notice that the integrated function depends only on $\vec{k}$ (and the measure we are considering is defined on three-momenta). We have formally extended it to be a function of $\vec{k}$, k_0 but this is only a trick because the δ function forces the variables to be related by the *mass shell condition* $k^2 = m^2$.
The form of the measure we got allows us to show immediately the invariance under a Lorentz transformations.

- d^4k is invariant since the jacobian determinant of the change of variables is 1.
- $\delta(k^2 - m^2)$ is a function of the scalar $k^2 = k_\mu k^\mu$ and therefore it's itself invariant.
- The theta function is not a priori invariant under Lorentz transformations: it is so only if the sign of k'_0 is the same of that of k_0. Lorentz transformations in general don't preserve the sign of the 0-component of a four vector: for example if $v_\mu = (1, 0, 0, 2)$, then one can easily find a boost in the third direction that makes $v'_0 < 0$:
$$v'_0 = \gamma v_0 - \beta\gamma v_3 = (1 - 2\beta)\gamma \implies \beta \geq \frac{1}{2}.$$
However one has to recall that the mass-shell condition $k_0 = \sqrt{|\vec{k}|^2 + m^2}$ makes k_μ a timelike fourvector (a fourvector in which the 0-component is larger than the modulus of its spatial threevector), that is to say a vector which lies inside the future lightcone centered in the origin. Transformations of the orthochronous Lorentz group, defined by the condition $\Lambda^0{}_0 > 0$ (those non containing the time reversal) send the future lightcone in itself and therefore if the four vector k_μ have positive 0-component, the same will be for k'_μ.
One can prove this explicitly. The transformed 0-component of k^μ is $k'^0 = \Lambda^0{}_0 k^0 + \sum_i \Lambda^i{}_0 k^i$, with $k^0 \geq \sqrt{\sum_i (k^i)^2}$, due to the mass-shell condition. The defining relation $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ implies $\Lambda^0{}_0 > \sqrt{\sum_i (\Lambda^i{}_0)^2}$. Moreover, since $\sum_i \Lambda^i{}_0 k^i \geq -\sqrt{\sum_i (k^i)^2}\sqrt{\sum_i (\Lambda^i{}_0)^2}$, then:
$$(k^0)' \geq \Lambda^0{}_0 k^0 - \sqrt{\sum_i (k^i)^2}\sqrt{\sum_i (\Lambda^i{}_0)^2} \geq \left(\Lambda^0{}_0 - \sqrt{\sum_i (\Lambda^i{}_0)^2}\right) k^0 > 0$$
implying that the sign of k'^0 is the same as the one of k^0.
Therefore the theta function will be left invariant in the distribution (1).

One can also check it explicitly performing Lorentz transformations: clearly the measure $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant under space rotations since d^3k is so and k_0 is a scalar under $SO(3)$. Therefore only pure boosts are left to check. Consider then a boost in the direction $\vec{n}$ with rapidity η (recall that the rapidity is defined as $\eta = \tanh^{-1}(\beta)$): we decompose the spatial momentum $\vec{k}$ in its longitudinal (i.e. along $\vec{n}$) and transverse (i.e. orthogonal to $\vec{n}$) parts: $\vec{k} = \vec{k}_T + \vec{k}_L$. Then the transformed quantities are:

$$\begin{aligned} k'_0 &= k_0 \cosh(\eta) + |\vec{k}_L| \sinh(\eta), \\ \vec{k}'_L &= k_0 \vec{n} \sinh(\eta) + \vec{k}_L \cosh(\eta), \\ \vec{k}'_T &= \vec{k}_T. \end{aligned}$$

Note that the direction of $\vec{k}_L$ is fixed to be parallel to $\vec{n}$, therefore one can remove the symbol of vector and consider only the modulus k_L. One can as well decompose the differential $d^3k \longrightarrow d^2k_T\, dk_L$; therefore the measure transforms as:

$$\begin{aligned} d^2k'_T &= d^2k_T, \\ dk'_L &= \frac{\partial k'_L}{\partial k_L} dk_L = \frac{\partial}{\partial k_L}(k_0 \sinh(\eta) + k_L \cosh(\eta)) dk_L = \frac{\partial}{\partial k_L}(\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}\ \sinh(\eta) + k_L \cosh(\eta)) dk_L \\ &= \left(\frac{k_L}{\sqrt{k_L^2 + |\vec{k}_T|^2 + m^2}} \sinh(\eta) + \cosh(\eta)\right) dk_L = \frac{1}{k_0}\left(k_L \sinh\eta + k_0 \cosh\eta\right) dk_L = \frac{k'_0}{k_0} dk_L. \end{aligned}$$

At the end the ratio $\frac{d^3k}{(2\pi)^3 2k_0}$ is invariant also under Lorentz boosts.

Finally one can consider the distribution $d^3k\delta^3(\vec{k})$. The fastest way to see that it is invariant is to see it as the result of an integration over k_0:

$$d^3k\delta^3(\vec{k}) = \int dk^0 d^3k\, \delta^4(k) = \int d^4k\, \delta^4(k) \tag{2}$$

The expression on the right-hand side is Lorentz invariant because $d^4k' = |J(\Lambda)|d^4k$ and $\delta^4(k') = \delta^4(k)|J(\Lambda)|^{-1}$, where $J(\Lambda)$ is the determinant of the Jacobian of the Lorentz transformation.
Finally, the distribution $(2\pi)^3k^0\delta^3(\vec{k})$ is Lorentz-invariant because it's obtained by dividing the two invariant distributions $d^3k\delta^3(\vec{k})$ and $\frac{d^3k}{(2\pi)^3k^0}$.

Exercise 5

- Let $\pi^a(x) \to \pi'^a(x) = \pi^a(x) + i\alpha^i\Theta^a_{ib}\pi^b(x)$. We can now fix Θ^a_{ib} using the Poisson bracket, $\{\pi_a(x), \phi_b(y)\} = \delta_{ab}\delta^3(x-y)$.

$$\begin{aligned}\{\pi^a(x), \phi_b(y)\} \to \{\pi'^a(x), \phi'_b(y)\} &= \{\pi^a(x), \phi_b(y)\} + i\alpha^i\Theta^a_{ic}\{\pi^c(x), \phi_b(y)\} + i\alpha^iT^b_{ic}\{\pi^a(x), \phi_c(y)\} &(3)\\ &= \{\pi^a(x), \phi_b(y)\} + i\alpha^i(\Theta^a_{ib} + T^b_{ia})\delta^3(x-y). &(4)\end{aligned}$$

  Since the Poisson bracket is invariant (it is a number) we must have $\Theta^a_{ib} = -T^b_{ia}$.

  Recall from exercise 3 of last set,

$$Q_i = \int d^3x\ J^0_i = \int d^3x\left(\pi^a\,\Delta_{ai} - \epsilon^0_i\mathcal{L}\right) = iT^b_{ia}\int d^3x\,\pi^a\,\phi_b \tag{5}$$

  where now $\Delta_{ai} = iT^b_{ia}\phi_b$ and $\epsilon^0_i = 0$.

  Therefore, we have

$$\{Q_i, \pi^a(x)\} = iT^c_{ib}\int d^3y\,\pi^b(y)\,\{\phi_c(y), \pi^a(x)\} = -iT^a_{ib}\pi^b(x) \tag{6}$$

  where we made use of $\{\pi^a(x), \phi_b(y)\} = \delta_{ab}\delta^3(x-y)$ in the last step.

- The Jacobi identity for ϕ_a, Q_i and Q_j reads

$$\{\phi_a, \{Q_i, Q_j\}\} + \{Q_i, \{Q_j, \phi_a\}\} + \{Q_j, \{\phi_a, Q_i\}\} = 0 \tag{7}$$

  we now use $\{Q_i, \phi_a\} = iT^b_{ia}\phi_b$ on the second and third terms to get

$$\{\phi_a, \{Q_i, Q_j\}\} = (T^c_{ib}T^b_{ja} - T^c_{jb}T^b_{ia})\phi_c = [T_i, T_j]^c_a\phi_c = if_{ijk}T^c_{ka}\phi_c = f_{ijk}\{Q_k, \phi_a\} = -f_{ijk}\{\phi_a, Q_k\} \tag{8}$$

  which for arbitrary ϕ_a implies

$$\{Q_i, Q_j\} = -f_{ijk}Q_k. \tag{9}$$

- We can now confirm the above result by computing the Poisson bracket $\{Q_i, Q_j\}$ explicitly. Using (5) we find

$$\frac{\delta Q_i}{\delta\pi^a(z)} = iT^c_{ia}\phi_c(z), \qquad \frac{\delta Q_j}{\delta\phi_a(z)} = iT^a_{jb}\pi^b(z), \tag{10}$$

  and, similarly, we find

$$\{Q_i, Q_j\} = \int d^3z\left[\frac{\delta Q_i}{\delta\pi^a(z)}\frac{\delta Q_j}{\delta\phi_a(z)} - \frac{\delta Q_j}{\delta\pi^a(z)}\frac{\delta Q_i}{\delta\phi_a(z)}\right] = -[T_i, T_j]^c_b\int d^3z\,\phi_c(z)\pi^b(z) = -f_{ijk}Q_k. \tag{11}$$

Exercise 6

Notice that this exercise is covered in Sec. 3.3.1 of the lecture notes.

Under a transformation parametrized by Lie parameter $\alpha_i(x)$, the coordinates and field change (to linear order in $\alpha_i(x)$) according to

$$x^\mu \longrightarrow x'^\mu = x^\mu - \alpha^i(x)\epsilon^\mu_i(x), \tag{12}$$

$$\phi_a(x) \longrightarrow \phi'_a(x) = \phi_a(x) + \alpha^i(x)\Delta_{ai}(x), \tag{13}$$

where ϵ_i^μ and Δ_{ai} depend on the specific transformation. The difference with respect to what was done in class is that $\alpha_i = \alpha_i(x)$ is local.

We now compute the variation of the action due to the local transformation. To linear order we have

$$\begin{aligned} S' &= \int d^4x' \mathcal{L}\left(\phi'_a(x'), \partial'_\mu \phi'_a(x')\right) && (14) \\ &= \int d^4x \left[1 - \partial_\mu(\alpha^i \epsilon_i^\mu)\right] \left[\mathcal{L}\left(\phi'_a(x), \partial_\mu \phi'_a(x)\right) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi'_a(x), \partial_\mu \phi'_a(x)\right)\right] && (15) \\ &= \int d^4x \left[1 - \partial_\mu(\alpha^i \epsilon_i^\mu)\right] \left[\mathcal{L}\left(\phi'_a(x), \partial_\mu \phi'_a(x)\right) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi_a(x), \partial_\mu \phi_a(x)\right)\right] && (16) \end{aligned}$$

where in the last equality we made use of the fact that $\alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi'_a(x), \partial_\mu \phi'_a(x)\right) = \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi_a(x), \partial_\mu \phi_a(x)\right)$ to leading order in α_i.

So for we only used the change of the coordinates, eq. (12). We now make use of the change of the field, eq. (13). We have (again to linear order)

$$\mathcal{L}\left(\phi'_a(x), \partial_\mu \phi'_a(x)\right) = \mathcal{L}\left(\phi_a(x), \partial_\mu \phi_a(x)\right) + \alpha^i(x)\Delta_{ai}(x) \frac{\partial \mathcal{L}}{\partial \phi_a} + \partial_\mu(\alpha^i(x)\Delta_{ai}(x)) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}. \tag{17}$$

Plugging into (16) we get (at linear order)

$$S' = \int d^4x \left[\left(1 - \partial_\mu(\alpha^i \epsilon_i^\mu)\right) \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right) + \alpha^i \Delta_{ai} \frac{\partial \mathcal{L}}{\partial \phi_a} + \partial_\mu(\alpha^i \Delta_{ai}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right] \tag{18}$$

where for notational purposes we did not write the argument over x. At this point it should be clear that every expression in the integrand is evaluated at x, in particular $\phi_a \equiv \phi_a(x)$ and $\alpha_i \equiv \alpha_i(x)$.

The variation of the action then reads

$$\begin{aligned} \delta S = S' - S &= \int d^4x \left[-\partial_\mu(\alpha^i \epsilon_i^\mu)\mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right) - \alpha^i \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right) + \alpha^i \Delta_{ai} \frac{\partial \mathcal{L}}{\partial \phi_a} + \partial_\mu(\alpha^i \Delta_{ai}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right] && (19) \\ &= \int d^4x \left[\alpha^i \left[\Delta_{ai} \frac{\partial \mathcal{L}}{\partial \phi_a} + \partial_\mu \Delta_{ai} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \epsilon_i^\mu \partial_\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right) - \partial_\mu \epsilon_i^\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right)\right]\right. && (20) \\ &\qquad \left. + \ \partial_\mu \alpha^i \left[\Delta_{ai} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \epsilon_i^\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right)\right]\right], && (21) \end{aligned}$$

where we separated terms proportional to α^i from terms proportional to $\partial_\mu \alpha^i$. This is because we are told that the global transformation is a symmetry, i.e. $\delta S = 0$ for constant α^i. In this case the second term drops, because $\partial_\mu \alpha^i = 0$, and if $\delta S = 0$ we have to conclude that the integral over the first term is zero. As with the derivation of Noether's theorem (see lecture notes) the integration region is arbitrary, therefore, not only the integral but the integrand itself must vanish.

We conclude that even if α^i is taken to depend on space-time point, only the second term contributes and we have

$$\delta S = \int d^4x \, \partial_\mu \alpha^i J_i^\mu \tag{22}$$

where $J_i^\mu = \Delta_{ai} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \epsilon_i^\mu \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right)$ is the Noether current associated with the global symmetry for which $\alpha^i = \text{constant}$.

This is a good way to obtain the Noether current associated with the symmetry. Let us consider a complex scalar field charged under $U(1)$. The Lagrangian reads

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi. \tag{23}$$

Under the $U(1)$ we have

$$\phi \to e^{i\alpha(x)}\phi \approx \phi + i\alpha(x)\phi, \qquad \phi^\dagger \to e^{-i\alpha(x)}\phi^\dagger \approx \phi^\dagger - i\alpha(x)\phi^\dagger, \tag{24}$$

so that $\Delta_\phi = i\phi$ and $\Delta_{\phi^\dagger} = -i\phi^\dagger$ and $\epsilon^\mu = 0$.

The Lagrangian will change as

$$\begin{aligned}\mathcal{L} \to \mathcal{L}' &= \partial_\mu \phi^\dagger \partial^\mu \phi + \partial_\mu(-i\alpha\phi^\dagger)\partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (i\alpha\phi) - m^2 \phi^\dagger \phi - m^2(i\alpha\phi\phi^\dagger - i\alpha\phi\phi^\dagger) \qquad (25)\\ &= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \partial_\mu \alpha \left[-i\phi^\dagger(\partial^\mu \phi) + i(\partial^\mu \phi^\dagger)\phi\right] = \mathcal{L} + \partial_\mu \alpha J^\mu \qquad (26)\end{aligned}$$

with

$$J^\mu = i\left[(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right]. \qquad (27)$$