Quantum Field Theory

Set 6: solutions

Exercise 1

- If $V_{ij} = v^a X^a_{ij}$ then (ommitting the matrix indices)

$$V' = D(g)VD(g^{-1}) = v^a\left[D(g)X^aD(g^{-1})\right] = v^a\left[e^{iX^b\alpha^b}X^ae^{-iX^c\alpha^c}\right]. \tag{1}$$

  We now use the Hadamard formula to write

$$e^{iX^b\alpha^b}X^ae^{-iX^c\alpha^c} = X^a + i\alpha^b[X^b, X^a] - \frac{\alpha^b\alpha^c}{2!}[X^b,[X^c,X^a]] + \dots \tag{2}$$

$$= X^a + \alpha^b f^{abc}X^c + \frac{\alpha^b\alpha^c}{2!}f^{cad}f^{bde}X^e + \dots \tag{3}$$

  where use was made of the Lie algebra.

  Note that every term in the series is a nested commutator which, due to the Lie algebra, will always end up in a single generator, as shown above for the first few terms.

  We can then write

$$V' = \left[v^a + \alpha^b f^{cba}v^c + \frac{\alpha^b\alpha^c}{2}f^{ced}f^{bda}v^e + \dots\right]X^a = v'^a X^a, \tag{4}$$

  where we swapped some indices to factor out X^a.

  This shows that $R(g)$ maps the vector space spanned by the generators to itself.

- Since $R(g)$ acts on the generators, then its dimension is the number of generators N, i.e. the dimension of the Lie group. In order for $R(g)$ to indeed define a representation it must faithfully *represent* the group elements (and their properties) on a vector space. Since $R(g)$ acts through left and right multiplication of $D(g)$ which is a generic representation we conclude that $R(g)$ is also a representation.

  Concretely,

  1. *Identity element.*

$$R(e) : V \to V' = D(e)VD(e^{-1}) = V. \implies R(e) = 1. \tag{5}$$

  2. *Inverse element.*

$$R(g^{-1}) : V \to V' = D(g^{-1})VD(g) = [D(g)]^{-1}V[D(g^{-1})]^{-1}. \implies R(g^{-1}) = R(g)^{-1}. \tag{6}$$

  3. *Group product.*

$$R(g_1)R(g_2) : V \to V' = D(g_1)D(g_2)VD(g_2^{-1})D(g_1^{-1}) = D(g_1g_2)VD(g_2^{-1}g_1^{-1}) \tag{7}$$

$$= D(g_1g_2)VD((g_1g_2)^{-1}). \implies R(g_1)R(g_2) = R(g_1g_2). \tag{8}$$

- Given $v'^i = R^{ij}(g)v^j$ and $R(g(\alpha)) = e^{i\tilde{X}^a\alpha^a}$ we find to linear order

$$R^{ij} = \delta^{ij} + i\tilde{X}^a_{ij}\alpha^a + \dots \tag{9}$$

  From eq. (4) we have

$$v'^i = \left[\delta^{ij} + \alpha^a f^{aij} + \dots\right]v^j \tag{10}$$

  where we swapped the following indices in eq. (4): $(a,b,c) \to (i,a,j)$ and used the cyclic property $f^{jai} = f^{aij}$.

  Comparing the above two equations we read off

$$\tilde{X}^a_{ij} = -if^{aij}. \tag{11}$$

- We have already showed that $R(g)$ is a representation. Therefore, its generators $\tilde{X}$ must satisfy the Lie algebra. Let's now check how this comes about explicitly.

  We have
$$[\tilde{X}^a, \tilde{X}^b]_{ij} = \tilde{X}^a_{ic}\tilde{X}^b_{cj} - \tilde{X}^b_{ic}\tilde{X}^a_{cj} = -f^{aic}f^{bcj} + f^{bic}f^{acj} = -f^{aci}f^{bjc} - f^{bci}f^{jac}. \tag{12}$$
  We now use the Jacobi identity
$$f^{aci}f^{bjc} + f^{bci}f^{jac} + f^{jci}f^{abc} = 0 \tag{13}$$
  to get
$$[\tilde{X}^a, \tilde{X}^b]_{ij} = f^{jci}f^{abc} = f^{abc}f^{cij} = if^{abc}\left(-if^{cij}\right) = if^{abc}\tilde{X}^c_{ij}. \tag{14}$$

Exercise 2

Consider first $j = 1$. We have:

$$T^3\left|1\right\rangle = \left|1\right\rangle, \quad T^3\left|0\right\rangle = 0, \quad T^3\left|-1\right\rangle = -\left|-1\right\rangle,$$
$$T^+\left|1\right\rangle = 0, \qquad T^+\left|0\right\rangle = \left|1\right\rangle, \qquad T^+\left|-1\right\rangle = \left|0\right\rangle.$$

We represent the states $|m\rangle$ as vector in $\mathbb{R}^3$ in the simplest way:

$$\left|1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \left|0\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \left|-1\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To construct the matrices of the generators then we just compute their components. For instance for T^3

$$(T^3)_{mm'} \equiv \langle m|T^3|m'\rangle = \delta_{mm'}m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The others are computed similarly:

$$T^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^- = (T^+)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\Longrightarrow \quad T^1 = \frac{T^+ + T^-}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = i\frac{T^- - T^+}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

The same can be used for $j = 3/2$ and $j = 2$. For $j = 3/2$ we obtain:

$$T^3 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}, \qquad T^+ = \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = (T^-)^\dagger.$$

For J=2:

$$T^3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \qquad T^+ = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (T^-)^\dagger.$$

Since T^3 is diagonal in this basis, we find $(\exp[i\phi(T^3)])_{mm'} = e^{im\phi}\delta_{mm'}$:

$$j = 1 \quad \longrightarrow \quad e^{i\phi T^3} = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix},$$

$$j = 3/2 \quad \longrightarrow \quad e^{i\phi T^3} = \begin{pmatrix} e^{i\frac{3}{2}\phi} & 0 & 0 & 0 \\ 0 & e^{i\frac{1}{2}\phi} & 0 & 0 \\ 0 & 0 & e^{-i\frac{1}{2}\phi} & 0 \\ 0 & 0 & 0 & e^{-i\frac{3}{2}\phi} \end{pmatrix},$$

$$j = 3/2 \quad \longrightarrow \quad e^{i\phi T^3} = \begin{pmatrix} e^{i2\phi} & 0 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\phi} & 0 \\ 0 & 0 & 0 & 0 & e^{-i2\phi} \end{pmatrix}.$$

Exercise 3

The explicit form of the three matrices is:

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad T^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \qquad T^3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The group $SO(3)$ is defined as

$$SO(3) = \{R \in GL(3, \mathbb{R}) |\ RR^T = R^T R = 1,\ \det(R) = 1\}$$

Parametrizing a general element of the group using the exponential function, $R(\alpha) = e^{i\alpha^a T^a}$, one can translate the constraints on the elements of the group to constraints on the generators:

$$1 = RR^T = (1 + i\alpha^a T^a)(1 + i\alpha^b (T^b)^T) + O(\alpha^2) \implies T^a = -(T^a)^T.$$

The Algebra of $SO(3)$ is a vector space generated by $3(3-1)/2 = 3$ antisymmetric objects, together with the usual commutator $[,]$. The three matrices defined at the beginning are

- antisymmetric,
- independent,
- in number equal to the dimension of the space.

Therefore they form a basis for (a representation of) the algebra $so(3)$. Having an explicit representation of the generators of a Lie Algebra, one can compute the commutators between them and extract the structure constants. The commutation relations which one obtains in this way are the same as in all the other representations, since the structure of the algebra of course doesn't depend on its explicit representation.
In the present case one has

$$\begin{aligned} [T^1, T^2] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = iT^3. \end{aligned}$$

Similarly one can explicitly compute

$$[T^2, T^3] = iT^1, \qquad [T^1, T^3] = -iT^2,$$

and identify the structure constant of the group $f^{abc} = \epsilon_{abc}$. This is the Algebra of the angular momentum one is used to deal with for example in quantum mechanics. The statement that a state $|s\rangle$ has angular momentum J means that it belongs to a vector space on which acts a representation of the rotation group $SO(3)$ (call this representation j – we will see that representations can be labelled by an integer number). Under the action of the

group, $|s\rangle$ transforms according to $|s\rangle \to e^{i\alpha_a T_a^{(j)}}|s\rangle$, where $T_a^{(j)}$ are the generators of $SO(3)$ in the representation j.
Coming back to structure constants, it is also possible to extract the commutation relations using the implicit form $(T^a)_i^j = -i\epsilon_{aij}$:

$$\begin{aligned}\left[T^a, T^b\right]_i^k &= (T^a)_i^j (T^b)_j^k - (T^b)_i^j (T^a)_j^k = (-i)^2 \epsilon_{aij}\epsilon_{bjk} - (-i)^2 \epsilon_{bij}\epsilon_{ajk} \\ &= \epsilon_{abc}\epsilon_{cik} = i\epsilon_{abc}(T^c)_i^k,\end{aligned}$$

where the last equality is a consequence of the identity $\epsilon_{aij}\epsilon_{bjk} + \epsilon_{ajk}\epsilon_{bji} + \epsilon_{abj}\epsilon_{jik} = 0$ (which in the end is the Jacobi identity for the structure constants of $so(3)$).
One can show that a general element of the group $SO(3)$ is a rotation acting on three dimensional vectors. To see this one can consider the *fundamental (or defining) representation*, that is to say the explicit representation of the group $SO(3)$ on $\mathbb{R}^3$ that we have previously recalled. An element of the group depends on three parameters α^a: one can collect them in a vector and call $\vec{n} = \vec{\alpha}/|\vec{\alpha}|$ the direction of this vector and $\theta = |\vec{\alpha}|$ the modulus of the vector. It's easy to prove that the action of the element $R(\alpha) = e^{i\alpha^a T^a}$ on a vector $\vec{x}$ corresponds to a rotation of this vector of an angle θ around the direction $\vec{n}$. One can firstly consider an infinitesimal rotation ($\theta << 1$)

$$\begin{aligned}& R(\alpha)_i^j\, x_j \simeq \left(1 + i\theta n^a T^a + O(\alpha^2)\right)_i^j\, x_j \simeq \left(\delta_i^j + i\theta n^a (T^a)_i^j +\right)\, x_j = x_i + \theta\epsilon_{aij} n^a x_j \\ & \Longrightarrow R(\alpha) : \;\; \vec{x} \longrightarrow \vec{x} + \theta\, \vec{x} \wedge \vec{n}.\end{aligned}$$

One can verify that this is in accord with the usual way of representing a rotation: for example a rotation around the 3^{rd} direction by an angle θ produces a change in the $1,2$ plane according to

$$\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) \longrightarrow \left(\begin{array}{c} x_1\cos\theta + x_2\sin\theta \\ x_2\cos\theta - x_1\sin\theta \\ x_3 \end{array}\right) \simeq \left(\begin{array}{c} x_1 + x_2\theta \\ x_2 - x_1\theta \\ x_3 \end{array}\right) = \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) + \theta \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) \wedge \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right),$$

where we have expanded the trigonometric functions for small angles.
One can do more: exponentiating the generators one can obtain the explicit form of an element of $SO(3)$ and compare it with a generic finite rotation. It's particularly easy to perform this computation in the simple case where the rotation is around one of the axes: let's take again the 3^{rd} direction for concreteness. Recognizing that

$$(T^3)^{2n} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \equiv A,$$

then

$$\begin{aligned} R(\theta\vec{n}^3) &= e^{i\theta T^3} = 1 + i\theta T^3 - \frac{1}{2}\theta^2 (T^3)^2 + \dots \\ &= iT^3\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right) + A\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + 1 - A \\ &= \left[\begin{array}{ccc} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right].\end{aligned}$$

One immediately recognizes the usual form of a rotation by an angle θ in the $1-2$ plane.

Note. The group $SO(n)$, as well as other groups of linear transformations, is usually not defined in abstract by characterizing its elements g, but specifying the properties of one particular representation (the $fundamental$ or $defining$ representation): in the case of $SO(3)$ the fundamental representation contains the 3×3 orthogonal matrices with determinant $= 1$. This does not mean of course that the group has only that representation. For example, a quantity which is invariant under rotations transforms according to a one dimensional representation of $SO(3)$ in which the generators are identically $= 0$, while an object with angular momentum $j = 2$ transforms according to a five dimensional representation, i.e. a representation in which the transformations are represented by 5×5 matrices.

The rest of the exercise deals with another group, $SU(2)$, and the relation between this group and the group of rotations that we have analyzed in the first part. To begin with, one can recall the definition of the group as

$$SU(2) = \left\{U \in GL(2,\mathbb{C}) |\; UU^+ = U^+U = 1,\; \det(U) = 1\right\}.$$

Then one can consider the representation of the group acting on the vector space V defined to be:

$$V = \left\{ M \in M(2, \mathbb{C}) |\ M = M^{+},\ \mathrm{Tr}(M) = 0 \right\},$$

that is to say the set of hermitian traceless matrices. One can verify that this vector space coincides with the one that defines the Lie Algebra of $SU(2)$. Indeed for infinitesimal transformations

$$\begin{aligned} &1 = U^{\dagger}U = (1 - i\alpha^{a}(T^{a})^{\dagger})(1 + i\alpha^{b}T^{b}) + O(\alpha^{2}) \implies T^{a} = (T^{a})^{\dagger}, \\ &1 = \det(e^{i\alpha T}) = e^{i\alpha \mathrm{Tr}(T)} \implies \mathrm{Tr}(T) = 0, \end{aligned}$$

therefore the two vector spaces coincide. If one is able to find a basis of V this will also be a basis of the Lie Algebra of $SU(2)$. A basis of the vector space V is given for example by the three Pauli matrices:

$$\sigma^{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Having a basis of the Lie Algebra it's possible to compute the commutation relations as we did for $SO(3)$:

$$\begin{aligned} &\left[\sigma^{1}, \sigma^{2}\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma^{3}, \\ &\left[\sigma^{2}, \sigma^{3}\right] = 2i\sigma^{1}, \qquad \left[\sigma^{1}, \sigma^{3}\right] = -2i\sigma^{2}, \end{aligned}$$

therefore the matrices $\tau^{a} \equiv \sigma^{a}/2$ satisfy the algebra of $SU(2)$:

$$\left[\tau^{a}, \tau^{b}\right] = i\epsilon_{abc}\tau^{c},$$

which is exactly the same of that one of $SO(3)$. This is something that happens frequently: given a Lie Group one and only one Lie Algebra is associated to it, however the converse in not true; given a Lie Algebra there exists unique a connected and simply connected Lie group associated to it, but there may exist other different groups without these constraints associated to the same algebra.

To summarize, we are considering a representation of a Lie Group on its Lie Algebra; this particular representation is called *adjoint representation.* The action of an element U of the group on an element M of the space V is as follows:

$$U:\ M \longrightarrow M' = UMU^{\dagger}$$

The above action defines a good representation since

- It's a linear application from V to V; indeed $(M')^{\dagger} = M'$ and $\mathrm{Tr}(M') = \mathrm{Tr}(UMU^{\dagger}) = \mathrm{Tr}(M) = 0$.
- It respect the composition of the group transformations:

$$\begin{aligned} &U_{1}:\ M \longrightarrow M' = U_{1}MU_{1}^{\dagger}, \qquad U_{2}:\ M' \longrightarrow M'' = U_{2}M'U_{2}^{\dagger}, \\ &U_{2} \circ U_{1}: M \longrightarrow (U_{2} \circ U_{1})M(U_{2} \circ U_{1})^{\dagger} = U_{2}U_{1}MU_{1}^{\dagger}U_{2}^{\dagger} = M''. \end{aligned}$$

Any hermitian traceless matrix can be written as a linear combination of elements of the basis:

$$M = \begin{bmatrix} y_{3} & y_{1} - iy_{2} \\ y_{1} + iy_{2} & -y_{3} \end{bmatrix} = y_{i}\sigma^{i}.$$

From the above equality one can argue that an element M can be associated to a thee-dimensional vector $\vec{y} = (y_{1}, y_{2}, y_{3})$, which is the set of coordinates of the element M in the chosen basis. We know that a representation of a group is defined as a mapping between the group and the matrices acting on a vector space. After having chosen a basis one can also build the explicit form of the matrices associates to the element U of $SU(2)$. Here there is a scheme of the relations:

$$\begin{aligned} \Psi &: \text{Group} \longrightarrow \text{Matrices acting on V} \\ &: U \longrightarrow R_{i}^{j} \\ U &: V \longrightarrow V \\ &: M = y_{i}\sigma^{i} \longrightarrow UMU^{\dagger} = \tilde{y}_{i}\sigma^{i} \\ R &: V \longrightarrow V \\ &: y_{i} \longrightarrow \tilde{y}_{i} = R_{i}^{j}y_{j}. \end{aligned}$$

In order to get the form of the matrix R associated to a given element U one can consider an infinitesimal element of $SU(2)$ acting on M:

$$UMU^\dagger \simeq (1 + i\alpha^a \tau^a) y_i \sigma^i (1 - i\alpha^b \tau^b) = y_i \sigma^i + \frac{i}{2} \left[\sigma^a, \sigma^i\right] \alpha^a y^i + O(\alpha^2)$$
$$= y_i \sigma^i + i(i)\epsilon_{aic} \sigma^c \alpha^a y_i = (y^c - \epsilon_{cai} \alpha^a y_i) \sigma^c = \tilde{y}^c \sigma^c.$$

Therefore the matrix R_i^j associated to the element of the group U is a rotation of an angle $\theta = |\vec{\alpha}|$ around the direction identified by $\vec{\alpha}$. One has to notice an important feature of this relation: the element of the group U and $-U$ induce the same changing for the vector $\vec{y}$, therefore they have the same representative. The representation map is not injective, even if it's surjective.

To summarize, we have shown that the group $SU(2)$ and $SO(3)$ have the same Lie Algebra, even if they are different groups. This implies that given a representation of the Algebra one has for sure a representation of $SU(2)$ (because is connected and simply connected) but not necessarily a representation of the group $SO(3)$. It may happen however that some vector space support both the representations, as we have seen. In particular the adjoint representation of $SU(2)$ (the one on it's Lie Algebra that we have considered in this exercise) provides automatically a representation of $SO(3)$.

Exercise 4

Part 1

A spin 1 representation is made of three states $\{|1\rangle, |0\rangle, |-1\rangle\}$, on which the generators act as:

$$T^3 |1\rangle = |1\rangle, \quad T^3 |0\rangle = 0, \quad T^3 |-1\rangle = -|-1\rangle,$$
$$T^+ |1\rangle = 0, \qquad T^+ |0\rangle = |1\rangle, \qquad T^+ |-1\rangle = |0\rangle,$$
$$T^- |1\rangle = |0\rangle, \qquad T^- |0\rangle = |-1\rangle, \qquad T^+ |-1\rangle = 0.$$

Consider now the tensor product representation, i.e. the linear space formed by the vectors:

$$|m_i\rangle_{(1)} \otimes |m_j\rangle_{(2)} = |m_i; m_j\rangle_{(1)\otimes(2)}, \qquad m_i, m_j = -1, 0, 1.$$

Here the subscripts underline that the states belong to two different linear spaces. As showed in the previous set, generators in the tensor product representation are written as:

$$T^i_{(1)\otimes(2)} = T^i_{(1)} \otimes \mathbb{1}_{(2)} + \mathbb{1}_{(1)} \otimes T^i_{(2)}.$$

We omit the subscript $(1) \otimes (2)$ in the following. It follows immediately that T^3 is diagonal in the tensor product representation:

$$T^3 |m_1; m_2\rangle = (m_1 + m_2) |m_1; m_2\rangle \equiv M |m_1; m_2\rangle.$$

We can thus classify states according to their eigenvalue of T^3. We find:

- $M = 2$: $|1; 1\rangle$ forming a 1d vector space,
- $M = 1$: $\{|1; 0\rangle, |0; 1\rangle\}$ forming a 2d vector space,
- $M = 0$: $\{|0; 0\rangle, |1; -1\rangle, |-1; 1\rangle\}$ forming a 3d vector space,
- $M = -1$: $\{|-1; 0\rangle, |0; -1\rangle\}$ forming a 2d vector space,
- $M = -2$: $|-1; -1\rangle$ forming a 1d vector space.

The highest weight technique consists in taking the maximum M eigenvector and applying lowering operators to get a $J = M$ representation. In this case this is just the state $|1; 1\rangle$ with $M = 2$. Since there are no states with bigger M, this must be part of a $J = 2$ representation. We hence call

$$|2, 2\rangle \equiv |1; 1\rangle.$$

Here the notation $|J, M\rangle$ means spin J representation with eigenvalue M of T_3:

$$\vec{T}^2 |J, M\rangle = J(J+1) |J, M\rangle , \qquad T^3 |J, M\rangle = M |J, M\rangle .$$

The other vectors of the representation are obtained acting with T^-. For instance

$$T^- |2, 2\rangle = T^- |1; 1\rangle = |0; 1\rangle + |1; 0\rangle \quad \Longrightarrow \quad |2, 1\rangle = \frac{1}{\sqrt{2}} (|0; 1\rangle + |1; 0\rangle) .$$

The prefactor is obtained by requiring normalization. Iterating we obtain:

$$\begin{aligned} |2, 0\rangle &= \frac{1}{\sqrt{6}} (2 |0, 0\rangle + |1, -1\rangle + |-1, 1\rangle) , \\ |2, -1\rangle &= \frac{1}{\sqrt{2}} (|0; -1\rangle + |-1; 0\rangle) , \\ |2, -2\rangle &= |-1; -1\rangle . \end{aligned}$$

The representation of course stops here, indeed $T^- |2, -2\rangle = 0$. As expected, in the $J = 2$ there is exactly one vector for each M. We thus still need to understand how the remaining vectors organize themselves into representations of $SU(2)$.
In order to proceed, let us recall that in general a vector belonging to a representation J is an eigenvector of the Casimir operator with eigenvalue $J(J+1)$. Since the Casimir operator is diagonal and proportional to the identity in any irreducible representation it follows that two vectors belonging to different representations are orthogonal. We have arranged the only $M = 2$ state in a $J = 2$ representation. There is instead only one $M = 1$ vector (up to normalization) which is orthogonal to $|2, 1\rangle$, which is given by:

$$|1, 1\rangle \equiv \frac{1}{\sqrt{2}} (|1; 0\rangle - |0; 1\rangle) .$$

It is easy to verify $T^+ |1, 1\rangle = 0$, hence it must belong to a $J = 1$ representation. The other vectors in $J = 1$ are obtained acting with T^-:

$$\begin{aligned} |1, 0\rangle &= \frac{1}{\sqrt{2}} (|1; -1\rangle - |-1; 1\rangle) , \\ |1, -1\rangle &= \frac{1}{\sqrt{2}} (|0; -1\rangle - |-1; 0\rangle) . \end{aligned}$$

We used all $M \neq 0$ vectors at our disposal. The only $M = 0$ vector left which is orthogonal to all the others thus corresponds to a $J = 0$ trivial representation:

$$|0, 0\rangle = \frac{1}{\sqrt{3}} (|0; 0\rangle - |1; -1\rangle - |-1; 1\rangle) .$$

Finally using

$$\vec{T}^2_{(1)\otimes(2)} = \vec{T}^2_{(1)} \otimes \mathbb{1}_{(2)} + \mathbb{1}_{(1)} \otimes \vec{T}^2_{(2)} + \frac{1}{2} \left(T^+_{(1)} \otimes T^-_{(2)} + T^-_{(1)} \otimes T^+_{(2)} \right) + 2 T^3_{(1)} \otimes T^3_{(2)},$$

it is possible to check that the representations we constructed have the right eigenvalue of the Casimir.

Part 2

Consider the product of two vectors $v_i w_j$. Under rotations this product obviously transform as:

$$v_i w_j \longrightarrow \sum_{k,m} R_{ik} R_{jm} v_k w_m .$$

This is obviously a representations of $SU(2)$. Hence we define a 2-tensor T_{ij} as an object which transforms under rotation as

$$T_{ij} \longrightarrow \sum_{k,m} R_{ik} R_{jm} T_{km} .$$

Notice that if we think of T as a matrix, we can rewrite the transformation rule as:

$$T \longrightarrow RTR^T.$$

T_{ij} has 9 components, as the number of independent vectors in the previous part. We expect to be able to decompose its components in three different representations: $J = 0, 1, 2$.
A $J = 0$ rep. simply corresponds to a quantity which is unchanged by rotations. It is easy to verify that this is given by the trace:

$$Tr[T] \longrightarrow Tr[RTR^T] = Tr[TR^T R] = Tr[T].$$

Notice now that simmetricity property are not changed by rotations. Define

$$A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right),$$

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) - \delta_{ij} Tr[T],$$

$$T = S + A, \qquad S^T = S, \quad A^T = -A.$$

Then it is easy to check that also $S' = RSR^T = S'^T$ and $A' = RAR^T = -A'^T$. Summing up everything we found that the trace, the antisymmetric part and the traceless symmetric part of a tensor transform independently. We already saw that $Tr[T]$ is a scalar. Since S has 5 independent components and A has 3 independent components, these must form a spin 2 and a spin 1 representations.
Notice that to check that A_{ij} transform as a usual vector, you can define:

$$v_i = \sum_{kl} \epsilon_{ikl} A_{kl}.$$

Then using $\epsilon_{ijk} = \sum_{m,n,l} R_{im} R_{jn} R_{kl} \epsilon_{mnl}$, one can check that v_i transforms in the usual way.

Exercise 5

Lorentz transformations are defined as the linear transformations acting on the spacetime coordinates that leave invariant the spacetime distance

$$s^2 = c^2 t^2 - \vec{x} \cdot \vec{x} = x^\mu x^\nu \eta_{\mu\nu}.$$

If one applies such a transformation to the four-vector x^μ, namely $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$, and imposes this to leave invariant the above defined distance one gets the constraint

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}, \qquad \text{or} \qquad \Lambda^T \eta \Lambda = \eta.$$

This equation defines a relation between the set of 4×4 real matrices that identifies a group called

$$O(1,3) = \left\{ \Lambda \in GL(4, \mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \right\},$$

where $\eta = \text{diag}(1, -1, -1 - 1)$. Geometrically the Lorentz group corresponds to the set of transformations that preserve the generalized scalar product defined by the matrix η.
In order to identify the Lie algebra associated to the Lorentz group one can consider the infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + w^\mu{}_\nu$ and plug it inside the constraint:

$$\left(\delta^\mu{}_\rho + w^\mu{}_\rho\right) \eta_{\mu\nu} \left(\delta^\nu{}_\sigma + w^\nu{}_\sigma\right) = \eta_{\rho\sigma} + w^\mu{}_\rho \eta_{\mu\sigma} + \eta_{\nu\rho} w^\nu{}_\sigma + O(w^2) = \eta_{\rho\sigma}$$
$$\Longrightarrow \quad w_{\rho\sigma} = -w_{\sigma\rho},$$

therefore the algebra consist of the antisymmetric 4×4 real matrices and thus it has dimension $\frac{4\times3}{2} = 6$. One would like to write the general element of the algebra as a linear combination of generators $w^\mu{}_\nu = -i w^a (J^a)^\mu{}_\nu / 2$. In order to write a compact expression for the generators it's useful to make use of a different notation: instead of a single index $a = 1, 2, ..., 6$ one can use a pair of indices $\alpha, \beta = 0, 1, 2, 3$ and make the following identification:

a	$\alpha\,\beta$
1	01
2	02
3	03
4	12
5	13
6	23

with $\quad T^{\alpha\beta} = -T^{\beta\alpha}.$

In this way the pairs of *spacetime indices* label exactly six generators. Now one is able to write a complete basis for the Lie algebra:

$$\mathcal{B}: \; (\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma} = i\,(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma})\,,$$

where as already explained the indices inside the parenthesis label the six generators while the other two are the proper indices of the matrix. Just to make an example, the matrix $\mathcal{J}^{01}$ is of the form

$$(\mathcal{J}^{01})^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the generators $(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma}$ are not antisymmetric matrices: only the $(\mathcal{J}^{\mu\nu})^{\rho\sigma} \equiv i\,(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})$ are. Now that one has an explicit form for the generators it becomes possible to compute the commutation relations and read out the structure constants:

$$\begin{aligned}
\left([\mathcal{J}^{\mu\nu},\mathcal{J}^{\alpha\beta}]\right)^{\gamma}{}_{\rho} &= (\mathcal{J}^{\mu\nu})^{\gamma}{}_{\sigma}(\mathcal{J}^{\alpha\beta})^{\sigma}{}_{\rho} - (\mathcal{J}^{\alpha\beta})^{\gamma}{}_{\sigma}(\mathcal{J}^{\mu\nu})^{\sigma}{}_{\rho} \\
&= -\left(\eta^{\mu\gamma}\delta^{\nu}_{\sigma} - \eta^{\nu\gamma}\delta^{\mu}_{\sigma}\right)\left(\eta^{\alpha\sigma}\delta^{\beta}_{\rho} - \eta^{\beta\sigma}\delta^{\alpha}_{\rho}\right) + \begin{pmatrix} \mu \leftrightarrow \alpha \\ \nu \leftrightarrow \beta \end{pmatrix}.
\end{aligned}$$

The result of the commutator has be a matrix with indices $(\;)^{\gamma}{}_{\rho}$, therefore we try to reproduce this combination in the r.h.s of the above expression:

$$\begin{aligned}
\left([\mathcal{J}^{\mu\nu},\mathcal{J}^{\alpha\beta}]\right)^{\gamma}{}_{\rho} = &-\Bigg(\underbrace{\eta^{\mu\gamma}\eta^{\alpha\nu}\delta^{\beta}_{\rho}}_{1} - \underbrace{\eta^{\nu\beta}\eta^{\mu\gamma}\delta^{\alpha}_{\rho}}_{2} - \underbrace{\eta^{\mu\alpha}\eta^{\nu\gamma}\delta^{\beta}_{\rho}}_{3} + \underbrace{\eta^{\nu\gamma}\eta^{\beta\mu}\delta^{\alpha}_{\rho}}_{4}\Bigg) \\
&+\Bigg(\underbrace{\eta^{\alpha\gamma}\eta^{\mu\beta}\delta^{\nu}_{\rho}}_{4} - \underbrace{\eta^{\nu\beta}\eta^{\alpha\gamma}\delta^{\mu}_{\rho}}_{2} - \underbrace{\eta^{\mu\alpha}\eta^{\beta\gamma}\delta^{\nu}_{\rho}}_{3} + \underbrace{\eta^{\beta\gamma}\eta^{\nu\alpha}\delta^{\mu}_{\rho}}_{1}\Bigg) \\
= &\; i\left(\eta^{\nu\alpha}(\mathcal{J}^{\mu\beta})^{\gamma}{}_{\rho} - \eta^{\nu\beta}(\mathcal{J}^{\mu\alpha})^{\gamma}{}_{\rho} - \eta^{\mu\alpha}(\mathcal{J}^{\nu\beta})^{\gamma}{}_{\rho} + \eta^{\mu\beta}(\mathcal{J}^{\nu\alpha})^{\gamma}{}_{\rho}\right).
\end{aligned}$$

Summarizing

$$\left[\mathcal{J}^{\mu\nu},\mathcal{J}^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha}\right).$$

Let us come back to the usual notation in which generators are labelled by a single index and define

$$\begin{aligned}
&J^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk}, \qquad \mathcal{J}^{jk} = \epsilon^{jki}J^i, \\
&K^i = \mathcal{J}^{i0}, \\
&i,\, j,\, k = 1,2,3 \qquad \text{and} \qquad \epsilon^{123} = 1.
\end{aligned}$$

Note that J^i, K^i are still 4×4 matrices. One can rewrite the commutation relation in terms of the new quantities

$$\begin{aligned}
\left[J^i,J^j\right] &= \frac{1}{4}\epsilon^{iab}\epsilon^{jcd}\left[\mathcal{J}^{ab},\mathcal{J}^{cd}\right] = \frac{i}{4}\epsilon^{iab}\epsilon^{jcd}\left(\eta^{bc}\mathcal{J}^{ad} - \eta^{bd}\mathcal{J}^{ac} - \eta^{ac}\mathcal{J}^{bd} + \eta^{ad}\mathcal{J}^{bc}\right) \\
&= -\frac{i}{4}\epsilon^{iab}\epsilon^{jcd}\left(\delta^{bc}\mathcal{J}^{ad} - \delta^{bd}\mathcal{J}^{ac} - \delta^{ac}\mathcal{J}^{bd} + \delta^{ad}\mathcal{J}^{bc}\right) \\
&= -\frac{i}{4}\left(\epsilon^{iab}\epsilon^{jbd}\epsilon^{adk} - \epsilon^{iab}\epsilon^{jcb}\epsilon^{ack} - \epsilon^{iab}\epsilon^{jad}\epsilon^{bdk} + \epsilon^{iab}\epsilon^{jca}\epsilon^{bck}\right)J^k \\
&= i\left(\delta^{ij}\delta^{ad} - \delta^{id}\delta^{aj}\right)\epsilon^{adk}J^k = -i\epsilon^{jik}J^k, \\
\left[J^i,J^j\right] &= i\epsilon^{ijk}J^k.
\end{aligned}$$

One immediately recognizes the algebra of $SU(2)$: the above generators form a *subalgebra* of the Lorentz Algebra. Indeed the Lorentz group contains the spatial rotations as a *subgroup*. The other commutation relations read

$$\begin{aligned}
\left[J^i,K^j\right] &= \frac{1}{2}\epsilon^{ika}\left[\mathcal{J}^{ka},\mathcal{J}^{j0}\right] = \frac{i}{2}\epsilon^{ika}\left(\eta^{aj}\mathcal{J}^{k0} - \eta^{a0}\mathcal{J}^{kj} - \eta^{kj}\mathcal{J}^{a0} + \eta^{k0}\mathcal{J}^{aj}\right) \\
&= -\frac{i}{2}\epsilon^{ika}\left(\delta^{aj}\mathcal{J}^{k0} - \delta^{kj}\mathcal{J}^{a0}\right) = i\epsilon^{ijk}K^k, \\
\left[K^i,K^j\right] &= \left[\mathcal{J}^{i0},\mathcal{J}^{j0}\right] = i\left(\eta^{0j}\mathcal{J}^{i0} - \eta^{00}\mathcal{J}^{ij} - \eta^{ij}\mathcal{J}^{00} + \eta^{i0}\mathcal{J}^{0j}\right) = -i\mathcal{J}^{ij} = -i\epsilon^{ijk}J^k.
\end{aligned}$$

It's important to underline the commutation rules of the generators of *boosts* K^i with those of rotations J^i: it states that the generators of boosts transform under rotation as a vector, that is to say according to the representation $J = 1$ of $SU(2)$. This becomes evident if one considers the adjoin representation of the Lorentz group acting on its algebra.
The fact that the commutator of two K's is a J rather than another K can be guessed considering the parity transformations (i.e. transformation under reflection of spatial coordinates) of these vectors. The angular momentum J is invariant under parity (indeed it is the vector product of position and momentum, two polar vectors), while the boost generator K changes sign reflecting the coordinates: thus a product of two K's cannot be a proportional to a K, and since the algebra has to close, it cannot but be some linear combination of J's.

Exercise 6

The explicit expression for $\mathcal{J}^{10}$ is

$$\mathcal{J}^{10} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where in the right hand side every entry is understood to be a 2×2 block. In particular, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of the Pauli matrices, satisfying $\sigma_1^2 = 1_2$ (this can be shown by explicit computation or using in general the anticommutation relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$). It is now possible to write the Lorentz transformation as a Taylor expansion in η:

$$\begin{aligned} \Lambda &= 1_4 - i\eta\mathcal{J}^{10} + \frac{(-i\eta)^2}{2!}(\mathcal{J}^{10})^2 + \cdots \\ &= \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} - \eta \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{(-i\eta)^2}{2!} \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} + \cdots \\ &\equiv \begin{pmatrix} \lambda & 0 \\ 0 & 1_2 \end{pmatrix}, \end{aligned}$$

with

$$\lambda = 1_2 - \eta\sigma_1 + \frac{\eta^2}{2!}1_2 - \frac{\eta^3}{3!}\sigma_1 + \cdots .$$

Separating the terms proportional to 1_2 from the ones proportional to σ_1, and remembering the Taylor series $\cosh(x) = 1 + x^2/2! + \cdots$ and $\sinh(x) = x + x^3/3! + \cdots$, then one can rewrite λ as

$$\lambda = \cosh(\eta)1_2 - \sinh(\eta)\sigma_1 = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix},$$

which proves what required in the text. Moreover, given the standard form of a boost along x,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

one can immediately identify γ and β in terms of the rapidity as

$$\beta = \tanh(\eta), \qquad \gamma = \cosh(\eta).$$

For what concerns the composition of boosts one can write explicitely

$$\Lambda\Lambda' = \begin{pmatrix} \lambda & 0 \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} \lambda' & 0 \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} \lambda\lambda' & 0 \\ 0 & 1_2 \end{pmatrix},$$

where

$$\begin{aligned}
\lambda\lambda' &= \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} \cosh(\eta') & -\sinh(\eta') \\ -\sinh(\eta') & \cosh(\eta') \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\eta)\cosh(\eta') + \sinh(\eta)\sinh(\eta') & -\cosh(\eta)\sinh(\eta') - \sinh(\eta)\cosh(\eta') \\ -\cosh(\eta)\sinh(\eta') - \sinh(\eta)\cosh(\eta') & \cosh(\eta)\cosh(\eta') + \sinh(\eta)\sinh(\eta') \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\eta+\eta') & -\sinh(\eta+\eta') \\ -\sinh(\eta+\eta') & \cosh(\eta+\eta') \end{pmatrix},
\end{aligned}$$

and thus the composition of two boosts along x is a boost along x, its rapidity being the sum of rapidities of the two single boosts. These computations have been performed for the particular direction x, but can of course be extended to the other axes or to linear combination of them.
Note that the composition of rapidities could be proved without recursion to computations by considering that the total transformation is $\Lambda\Lambda' = \exp[-i\eta\mathcal{J}^{10}]\exp[-i\eta'\mathcal{J}^{10}] = \exp[-i(\eta+\eta')\mathcal{J}^{10}]$, which actually confirms the power and elegance of the exponential mapping.

More about $SU(2)$ and $SO(3)$

The Pauli matrices have many properties: in addition to the fact that they satisfy the algebra of $SU(2)$ we can easily show that they satisfy a different algebra, that involves the anticommutators of two matrices $\{A, B\} = AB + BA$. Indeed

$$\{\sigma^a, \sigma^b\} = 2\delta^{ab}.$$

as one can directly verify. The above relation is called *Clifford's Algebra*. Note that we are not claiming that any representation of the algebra of $SU(2)$ satisfy also the Clifford's one. This is only a peculiarity of Pauli matrices and therefore holds only when we consider the space of 2×2 hermitian traceless matrices, not general representations.
Using the commutator and anticommutator one can easily write the product of two Pauli matrices in terms of one:

$$\sigma^a\sigma^b = \frac{1}{2}\{\sigma^a\sigma^b\} + \frac{1}{2}[\sigma^a\sigma^b] = \delta^{ab} \times 1_2 + i\epsilon_{abc}\sigma^c.$$

The above expression allows one to exponentiate immediately an element of the $SU(2)$ algebra and get the explicit form of an element of the group:

$$\begin{aligned}
\frac{i^{2n}}{2^{2n}(2n)!}\alpha^{a_1}...\alpha^{a_{2n}}\,\sigma^{a_1}...\sigma^{a_{2n}} &= \frac{i^{2n}}{2^{2n}(2n)!}\alpha^{a_1}...\alpha^{a_{2n}}\,\sigma^{a_3}...\sigma^{a_{2n}}(\delta^{a_1a_2} \times 1_2 + i\epsilon_{a_1a_2c}\sigma^c) \\
&= \frac{i^{2n}|\vec{\alpha}|^2}{2^{2n}(2n)!}\alpha^{a_3}...\alpha^{a_{2n}}\,\sigma^{a_3}...\sigma^{a_{2n}} = \frac{i^{2n}|\vec{\alpha}|^{2n}}{2^{2n}(2n)!} \times 1_2, \\
\frac{i^{2n+1}}{2^{2n+1}(2n+1)!}\alpha^{a_1}...\alpha^{a_{2n+1}}\,\sigma^{a_1}...\sigma^{a_{2n+1}} &= \frac{i^{2n+1}|\vec{\alpha}|^{2n}}{2^{2n+1}(2n+1)!}\alpha^{a_{2n+1}}\sigma^{a_{2n+1}}.
\end{aligned}$$

Therefore an element of the group becomes

$$\begin{aligned}
&U(\alpha) = e^{i\alpha^a\sigma^a/2} = 1 + i\frac{\alpha^a}{2}\sigma^a - \frac{1}{8}\alpha^a\alpha^b\sigma^a\sigma^b + ... = 1_2 \times \left(1 - \frac{|\vec{\alpha}|^2}{4\cdot 2!} + \dots\right) + i\sigma^a\frac{\alpha^a}{|\vec{\alpha}|}\cdot\left(\frac{|\vec{\alpha}|}{2} - \frac{|\vec{\alpha}|^3}{8\cdot 3!}\dots\right) \\
&= \cos\left(\frac{|\vec{\alpha}|}{2}\right) \times 1_2 + in^a\sigma^a\sin\left(\frac{|\vec{\alpha}|}{2}\right) \equiv k_0 \times 1_2 + ik_i\sigma^i.
\end{aligned}$$

where n^a is the unitary vector pointing in the same direction as α^a. One can see that the general element of the group is a linear combination of the identity and of the Pauli matrices. The coefficients of the linear combination are not independent since they must respect the determinant constraint:

$$1 = \det\begin{bmatrix} k_0 + ik_3 & ik_1 + k_2 \\ ik_1 - k_2 & k_0 - ik_3 \end{bmatrix} = k_0^2 + k_1^2 + k_2^2 + k_3^2.$$

The above expression is the equation that defines the embedding of a 3-sphere into $\mathbb{R}^4$. This parametrization shows that the group $SU(2)$, thought of as a manifold, is equivalent to S^3, which is a connected simply connected manifold.

Coming back to the first exercise one should recall that (the defining representation of) the group $SO(3)$ coincides with the adjoint representation of $SU(2)$. This representation is not injective because it associates two distinct elements of $SU(2)$ (U and $-U$) to the same element of $SO(3)$ (we say that $SU(2)$ is the double covering of $SO(3)$). This means that in order to visualize $SO(3)$ as a manifold one can think about a sphere where we identify a point with the opposite one: $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \sim -(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. The manifold obtained is usually denoted as $\frac{S^3}{\mathbb{Z}_2}$. This manifold is locally equivalent to the sphere, in particular they have the same tangent space, and this reflects the fact that the Algebras of $SO(3)$ and $SU(2)$ are the same. However the identification of opposite points has a crucial global consequence: this manifold is not simply connected (recall that a connected space is said simply connected if any closed curve can be continuously shrunk to a point). To see this, imagine a curve starting at the North Pole and ending at the South Pole. Since the starting and ending points are identified this curve is *close*. The considered curve however cannot be shrunk to a point without opening it, because as soon as we move one of the Poles the curve stops to be closed. To summarize the relation between the two groups is

$$SO(3) = \frac{SU(2)}{\mathbb{Z}_2}.$$

For completeness we define the group $\mathbb{Z}_2$, which is the pair $\{-1, 1\}$ together with the usual multiplication.