Quantum Field Theory

Set 5: solutions

Exercise 1

The following groups are the most common groups one can deal with in theoretical physics.

- $$U(N) \equiv \left\{U \in GL(N,\mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N\right\}$$

  This is the group of $N \times N$ complex unitary matrices. Clearly the inverse corresponds to the hermitian conjugate. One can consider the associated algebra $u(N)$ and take a complete basis T^a of this vector space. Here T^a represents a basis of generators and the label a runs from 1 to dim(algebra). In order to identify the structure of the algebra one can make use of the exponential map to write a generic element U of the group in terms of the generator T^a and some coordinate α^a:

  $$U_\alpha = e^{i\alpha^a T^a} \simeq 1_N + i\alpha^a T^a + O(\alpha^2).$$

  The unitarity of U implies that

  $$1_N = U_\alpha U_\alpha^\dagger \simeq (1_N + i\alpha^a T^a)\left(1_N - i\alpha^a (T^a)^\dagger\right) \simeq 1_N + i\alpha^a T^a - i\alpha^a (T^a)^\dagger.$$

  Therefore the generators are all the matrices that satisfy $T = T^\dagger$, that is to say the hermitian $N \times N$ matrices. One can easily compute the dimension of this vector space counting the number of independent parameters appearing in a generic hermitian matrix.

  $$T_{ij} = (T^\dagger)_{ij} = (T_{ji})^* \implies \begin{cases} \text{Elements on the diagonal are real: } N \text{ components.} \\ \text{Elements symmetric w.r.t the diagonal are} \\ \text{complex conjugate: } N(N-1) \text{ components.} \end{cases}$$

  In the end the dimension of the algebra (equal to the dimension of the vector space of complex hermitian matrices) is N^2. A complete set of generators for the group $U(N)$ is given by a complete basis of the complex hermitian $N \times N$ matrices.

- $$SU(N) \equiv \left\{U \in GL(N,\mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N,\ \det(U) = 1\right\}.$$

  The latter group is similar to the previous one but with an additional constraint: if in $U(N)$ the determinant of a matrix satisfies $|\det(U)| = 1$, here we choose only $\det(U) = 1$. This corresponds to considering only the subgroup of $U(N)$ connected to the identity. The additional requirement can be translated to the algebra using the relation

  $$\det(e^A) = e^{\mathrm{Tr}[A]}.$$

  Therefore the algebra is now composed by complex hermitian *traceless* $N \times N$ matrices. The tracelessness constraint consists in only one relation between the components of an hermitian matrix T since one already knows that all diagonal elements are real. The dimension of the algebra is therefore:

  $$\dim(su(N)) = N^2 - 1.$$

- $$SO(N) \equiv \left\{R \in GL(N,\mathbb{R}) \middle| RR^T = R^T R = 1_N,\ \det(R) = 1\right\}.$$

  This is the group of orthogonal real $N \times N$ matrices. Still using the exponential map

  $$R_\alpha = e^{\alpha^a T^a} \simeq 1_N + \alpha^a T^a + O(\alpha^2).$$

This time it's better to define the generator without the i in the exponent: in this way, since R is real also the T^a are real instead of purely imaginary. The orthogonality implies:

$$1_N = R_\alpha R_\alpha^T \simeq (1_N + \alpha^a T^a)\left(1_N + \alpha^a (T^a)^T\right) \simeq 1_N + \alpha^a T^a + \alpha^a (T^a)^T,$$

that is to say the algebra is formed by antisymmetric real matrices. The tracelessness is automatically satisfied since antisymmetric matrices have all zero components in the diagonal. The number of components of such a matrix are $N(N-1)/2$, which corresponds to the dimension of the algebra $so(N)$.

- $$O(N) \equiv \left\{R \in GL(N,\mathbb{R}) \middle| RR^T = R^T R = 1_N\right\}.$$

  The structure of the algebra is the same as the previous one since the removed constraint has no implication at the algebra level. However the group is not the same: one can think about $O(N)$ as $SO(N)$ with additional parities that invert an odd number of coordinates. For example $O(3)$ can be thought as the rotation group $SO(3)$ together with the following matrices

  $$P_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

  The latter are discrete symmetries: composing a generic element of $SO(N)$ with one of these, one can generate the whole $O(N)$. Note that in this case the exponential map doesn't cover all the group since it's formed by several disconnected pieces: the one containing the identity is the subgroup $SO(N)$ and one can reach the others acting with the parities.

- $$SL(N,\mathbb{C}) \equiv \left\{V \in GL(N,\mathbb{C}) \middle| \det(V) = 1\right\}.$$

  This is the group of complex $N \times N$ matrices with unitary determinant. Using the exponential map one obtains the constraint for the algebra:

  $$\det(V) = 1 = e^{i\alpha^a \mathrm{Tr}[T^a]} \;\Rightarrow\; Tr[T^a] = 0.$$

  Since the tracelessness this time is a complex statement, it contains two independent constraints and the dimension of the algebra is

  $$\dim(sl(N,\mathbb{C})) = 2N^2 - 2 = 2(N^2 - 1).$$

Exercise 2

We now show how one can build an irreducible representation of the Algebra of $SU(2)$ and therefore also a representation of the Group. Given the commutation relations

$$\left[T^a, T^b\right] = i\epsilon_{abc} T^c,$$

one can compute the following

$$\begin{aligned}
\left[T^\pm, T^\pm\right] &= \frac{1}{2}\left[T^1 \pm iT^2, T^1 \pm iT^2\right] = \pm\frac{i}{2}\left[T^1, T^2\right] \pm \frac{i}{2}\left[T^2, T^1\right] = 0, \\
\left[T^+, T^-\right] &= \frac{1}{2}\left[T^1 + iT^2, T^1 - iT^2\right] = -\frac{i}{2}\left[T^1, T^2\right] + \frac{i}{2}\left[T^2, T^1\right] = T^3, \\
\left[T^3, T^\pm\right] &= \frac{1}{\sqrt{2}}\left[T^3, T^1 \pm iT^2\right] = \frac{1}{\sqrt{2}}\left[T^3, T^1\right] \pm \frac{i}{\sqrt{2}}\left[T^3, T^2\right] = \frac{iT^2 \pm T^1}{\sqrt{2}} = \pm T^\pm.
\end{aligned}$$

It's easy to show that the sum of squared generators commutes with all the generators

$$\begin{aligned}
\left[\sum_{a=1}^{3} T^a T^a, T^b\right] &= \sum_{a=1}^{3}\left(T^a\left[T^a, T^b\right] + \left[T^a, T^b\right]T^a\right) = i\epsilon_{abc} T^a T^c + i\epsilon_{abc} T^c T^a \\
&= i\epsilon_{abc} T^a T^c - i\epsilon_{cba} T^c T^a = 0.
\end{aligned}$$

The operator $J^2 = \sum_{a=1}^{3} T^a T^a$ commutes with all the generators of the Algebra, therefore commutes with the whole Group. In an irreducible representation Ψ one can use the Schur's Lemma to prove that J^2 has to be a multiple of the identity:

$$\Psi : T^a \longrightarrow \tau^a \qquad \Psi : J^2 \longrightarrow \sum_{a=1}^{3} \tau^a \tau^a = \mu^2 \times 1,$$

where μ is some constant that we will determine in the following.
Let us consider an irreducible representation where generators are represented by $\tau^\pm$, τ^3, $\tau^a\tau^a = \mu^2 \times 1$, and let us consider inside the vector space an eigenvector $|m\rangle$ of the generator τ^3 relative to the eigenvalue m:

$$\tau^3 |m\rangle = m\,|m\rangle.$$

The action of one of the other generators $\tau^\pm$ sends $|m\rangle$ into another vector $|m'\rangle$ which one can show to be still an eigenvector of τ^3 but with a different eigenvalue:

$$\tau^3|m'\rangle = \tau^3\tau^\pm|m\rangle = \tau^\pm\tau^3|m\rangle + [\tau^3, \tau^\pm]|m\rangle = m\tau^\pm|m\rangle \pm \tau^\pm|m\rangle = (m \pm 1)\tau^\pm|m\rangle,$$

that is to say the $\tau^\pm$ generators acting on $|m\rangle$ change its eigenvalue by one unity. This is why they are called *raising and lowering operators*. More precisely, if we call $|m \pm 1\rangle$ the state normalized to one respect to a given scalar product, then

$$|\tau^\pm|m\rangle|^2 = \langle m|(\tau^\pm)^\dagger\tau^\pm|m\rangle = \frac{1}{2}\langle m|(\tau^1)^2 + (\tau^2)^2 \pm i[\tau^1, \tau^2]|m\rangle = \frac{1}{2}\langle m|\mu^2 - (\tau^3)^2 \mp \tau^3|m\rangle = \frac{1}{2}(\mu^2 - m(m \pm 1))$$

where it has been used $(\tau^\pm)^\dagger = \tau^\mp$. Therefore the correct normalization is

$$\tau^\pm|m\rangle = \frac{1}{\sqrt{2}}\sqrt{\mu^2 - m(m \pm 1)}|m \pm 1\rangle.$$

Moreover, from the previous equalities one can argue that $\mu^2 - m(m \pm 1) \geq 0$, since we deal with a space with positive definite norm ($|\tau^\pm|m\rangle|^2 \geq 0$). At the end

$$m^2 + |m| \leq \mu^2.$$

This statement has two important consequences: firstly it's a proof that μ^2 is a positive quantity, and secondly it imposes a limit on the dimension of an irreducible representation: indeed starting from a given state $|m-\rangle$ one can apply the raising operator to get another state, independent from the original one. This will increase also the value of m of one unity. If one were free to keep on applying τ^+ he would end with a violation of the inequality (note that since the Casimir operator $(\tau)^2$ is proportional to the identity, its eigenvalue μ^2 is constant, i.e. does not depend on m). Hence the action of the raising operator has to give a null state at a certain point. This happens only when $m(m + 1) = m_{max}(m_{max} + 1) = \mu^2$. Starting from the state $|m_{max}\rangle$ one can apply the lowering operator to decrease the value of m. As before after a finite number of steps one has to find a null state

$$(\tau^-)^{n+1}|m_{max}\rangle \propto \tau^-|m_{max} - n\rangle = 0 \qquad \text{for some } n,$$

and this will happen when $(m - n)(m - n - 1) = m_{min}(m_{min} - 1) = \mu^2$. Matching the two relations one finds

$$m_{min}(m_{min} - 1) = m_{max}(m_{max} + 1) \implies m_{max} = -m_{min}.$$

Moreover m_{min} has been obtained starting from m_{max} with an integer number of steps equal to $2m_{max} + 1$. This restricts the value of m_{max} to be a positive integer or semi-integer. Summarizing, using the notation $m_{max} = j$, an irreducible representation of the Algebra of $SU(2)$ is characterized by

- A vector space with dimension $2j + 1$ with a basis given by the eigenvectors of τ^3:

$$\{|m\rangle\}, \qquad -j \leq m \leq j.$$

- The generators on this vector space are represented as follows

$$\tau^3|m\rangle = m|m\rangle,$$
$$\sum_{a=1}^{3} \tau^a\tau^a|m\rangle = \mu^2|m\rangle = j(j + 1)|m\rangle,$$
$$\tau^\pm|m\rangle = \frac{1}{\sqrt{2}}\sqrt{j(j + 1) - m(m \pm 1)}|m \pm 1\rangle.$$

As already said, these are representation of the algebra and therefore also of the $SU(2)$ group. Not all of them are representations of $SO(3)$. The problem arises when one tries to pass from the algebra (which is somehow a local representation of the group) to a global representation of the group. $SO(3)$ has indeed the property that a rotation of 2π around any axis must coincide with the identity. This restricts the value of j to be only integer (we will see it explicitly in some example).

Finally one can consider some representation:

- $j = 0$ is the trivial representation and is called *scalar representation.*
- $j = 1/2$ is the first non trivial one. It's only a representation of $SU(2)$ and is called *spinorial representation.* It's composed by two states labelled by the value of j and m: $|j = 1/2, m = \pm 1/2\rangle$.
- $j = 1$ is a representation of both groups. It is called *vectorial representation* and corresponds to the adjoint of $SU(2)$ or the fundamental of $SO(3)$. A basis for this representation is given by three states labelled by

$$|1, 1\rangle,\ |1, 0\rangle,\ |1, -1\rangle.$$

Exercise 3

- By definition of direct sum we can write D in block diagonal form,

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

It then follows trivially that D is a representation (in matrix products of D's, D_1's and D_2's will never mix up, and since individually D_1 and D_2 are representations, so will D be).

It is also clear that the vector $v_1 \oplus v_2 = (v_1, v_2)$ has $\dim V_1 + \dim V_2$ components. So

$$\dim V_1 \oplus V_2 = \dim V_1 + \dim V_2.$$

For the final part of the question note that we can write A in blocks according to the V_1 and V_2 subspaces,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1}$$

By hypothesis the two matrices

$$AD = \begin{pmatrix} A_{11}D_1 & A_{12}D_2 \\ A_{21}D_1 & A_{22}D_2 \end{pmatrix}, \qquad DA = \begin{pmatrix} D_1A_{11} & D_1A_{12} \\ D_2A_{21} & D_2A_{22} \end{pmatrix}. \tag{2}$$

are equal. Given that D_1 and D_2 are inequivalent, the equality of the off-diagonal elements $A_{12}D_2 = D_1A_{12}$ and $A_{21}D_1 = D_2A_{21}$ imply, by the second Shur's lemma, $A_{12} = A_{21} = 0$. Given that D_1 and D_2 are irreducible, the equality of the diagonal elements $A_{11}D_1 = D_1A_{11}$, $A_{22}D_2 = D_2A_{22}$ imply, by the first Shur's lemma, that $A_{11} = \lambda_1 I$ and $A_{22} = \lambda_2 I$.

- Given two vector spaces V_1, V_2, with vectors $|v_1\rangle$, $|v_2\rangle$, the tensor product of the two is the set of all possible pairs:

$$V_1 \otimes V_2 = \{|v_1\rangle \otimes |v_2\rangle\ ,\ \text{where } v_i \in V_i\}\,.$$

Moreover, the tensor product is distributive,

$$(|v_1\rangle + |w_1\rangle) \otimes |v_2\rangle = |v_1\rangle \otimes |v_2\rangle + |w_1\rangle \otimes |v_2\rangle, \qquad |v_1\rangle \otimes (|v_2\rangle + \mid w_2\rangle) = |v_1\rangle \otimes |v_2\rangle + |v_1\rangle \otimes |w_2\rangle.$$

In addition it can be shown that a basis of the tensor product of two vector spaces is given by all the possible pairs obtained by taking one element from the basis of the first vector space and one element from the basis of the second vector space.

The representation acting on the tensor product space is called tensor product representation, and it is easy to show that indeed it is a true representation of the group (even if, in general, it is reducible). Denoting by

$D^1(g)$ and $D^2(g)$ two representations of the same element g of a given group $\mathcal{G}$, acting on vector spaces V_1 and V_2, the tensor product representation $D^1(g) \otimes D^2(g) \equiv D^{1\otimes 2}(g)$, has the following properties:

$$\begin{aligned}
D^{1\otimes 2}(g_a)D^{1\otimes 2}(g_b) &\equiv (D^1(g_a)\otimes D^2(g_a))(D^1(g_b)\otimes D^2(g_b)) = D^1(g_a)D^1(g_b)\otimes D^2(g_a)D^2(g_b) \\
&= D^1(g_a\circ g_b)\otimes D^2(g_a\circ g_b) = D^{1\otimes 2}(g_a\circ g_b), \\
D^{1\otimes 2}(e) &= D^1(e)\otimes D^2(e) = 1_{V_1}\otimes 1_{V_2} = 1_V,
\end{aligned}$$

where V is the tensor product space $V = V_1 \otimes V_2$.
In passing from first to second line it has been employed the fact that D^1 and D^2 act on *different* vector spaces, thus they commute (note that this is true even if D^1 and D^2 are two copies of *the same* representation). The system above shows that the tensor product representation is a representation of $\mathcal{G}$.
It is also possible to build explicitly the generators of the group in the tensor product representation. Denoting as $(t_1^a)_{ij}$ and $(t_2^a)_{xy}$ the generators in representations D^1 and D^2 respectively, one can write down the expression for an element near to the identity in representation $D^{1\otimes 2}$ as

$$\begin{aligned}
\left[D^1(\alpha)\right]_{ij}\left[D^2(\alpha)\right]_{xy} &= \left[D^{1\otimes 2}(\alpha)\right]_{ijxy} = [\delta_{ij} + i\alpha^a(t_1^a)_{ij}][\delta_{xy} + i\alpha^a(t_2^a)_{xy}] + O(a^2) \\
&= \delta_{ij}\delta_{xy} + i\alpha^a\left[(t_1^a)_{ij}\delta_{xy} + \delta_{ij}(t_2^a)_{xy}\right] + O(\alpha^2),
\end{aligned}$$

which can be written in tensor product notation as

$$\begin{aligned}
D^1(\alpha)\otimes D^2(\alpha) &= D^{1\otimes 2}(\alpha) = [1_{V_1} + i\alpha^a t_1^a]\otimes[1_{V_2} + i\alpha^a t_2^a] + O(a^2) \\
&= 1_V + i\alpha^a\left[t_1^a\otimes 1_{V_2} + 1_{V_1}\otimes t_2^a\right] + O(\alpha^2).
\end{aligned}$$

The operators in squared parentheses are the generators in the tensor product representation.

Exercise 4

A particle with spin j is an object that under rotations transforms as a state of the representation j of the group $SU(2)$. If one chooses a spatial direction, the 3rd one for example, the representation j can be defined considering the possible eigenvectors of the generator of rotation in this direction, τ^3. These eigenvectors form a basis $\mathcal{B}$ of the $2j+1$ dimensional vector space where the group is represented:

$$\mathcal{B} = \{|j,m\rangle,\ m = -j,\ -j+1, ..., j-1,\ j\}\,.$$

The action of the generators on this vector space is given by

$$\begin{aligned}
&\tau^3|j,m\rangle = m|j,m\rangle, \\
&\tau^\pm|j,m\rangle = \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m(m\pm 1)}|j,m\pm 1\rangle, \\
&\sum_{i=1}^{3}(\tau^i)^2|j,m\rangle = j(j+1)|j,m\rangle.
\end{aligned}$$

Let us specialize to the $j = 1/2$ representation. The vector space in this case is 2-dimensional and a basis consists simply of the two states

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle,\quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle,$$

and the generators are represented by the three matrices

$$\tau^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},\qquad \tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\qquad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider now two copies of the previous representation, corresponding for example to two distinct particles with spin 1/2 each. If one wants to consider the spin of the bound state formed by these two particles one should use the notion tensor product of two vector spaces introduced in exercise 3.

Here we are considering the tensor product of two 2-dimensional vector spaces on which the representation $j = 1/2$ of $SU(2)$ acts. A complete basis for the tensor product space is given by the set

$$\mathcal{B}_V = \left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle,\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle\ \right\}.$$

The tensor product is hence 4-dimensional. One can use the following notation for short:

$$\mathcal{B}_V = \{|\uparrow\uparrow\rangle\,,\ |\uparrow\downarrow\rangle\,,\ |\downarrow\uparrow\rangle\,,\ |\downarrow\downarrow\rangle\}\,.$$

From exercise 3 we know that the generators of the direct product are the 'sum' of the generators of each representation,

$$\begin{aligned}\tau_V^3 &= \tau^3 \otimes 1 + 1 \otimes \tau^3,\\ \tau_V^+ &= \tau^+ \otimes 1 + 1 \otimes \tau^+,\\ \tau_V^- &= \tau^- \otimes 1 + 1 \otimes \tau^-.\end{aligned}$$

One can verify that the elements of the basis of the tensor product space are still eigenvectors of the generator τ_V^3:

$$\begin{aligned}\tau_V^3\,|\uparrow\uparrow\rangle &= \tau^3\,|\uparrow\rangle \otimes 1\,|\uparrow\rangle + 1\,|\uparrow\rangle \otimes \tau^3\,|\uparrow\rangle = \frac{1}{2}\,|\uparrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes \frac{1}{2}\,|\uparrow\rangle = |\uparrow\uparrow\rangle\,,\\ \tau_V^3\,|\uparrow\downarrow\rangle &= \tau^3\,|\uparrow\rangle \otimes 1\,|\downarrow\rangle + 1\,|\uparrow\rangle \otimes \tau^3\,|\downarrow\rangle = \frac{1}{2}\,|\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes \frac{-1}{2}\,|\downarrow\rangle = 0,\\ \tau_V^3\,|\downarrow\uparrow\rangle &= \tau^3\,|\downarrow\rangle \otimes 1\,|\uparrow\rangle + 1\,|\downarrow\rangle \otimes \tau^3\,|\uparrow\rangle = \frac{-1}{2}\,|\downarrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes \frac{1}{2}\,|\uparrow\rangle = 0,\\ \tau_V^3\,|\downarrow\downarrow\rangle &= \tau^3\,|\downarrow\rangle \otimes 1\,|\downarrow\rangle + 1\,|\downarrow\rangle \otimes \tau^3\,|\downarrow\rangle = \frac{-1}{2}\,|\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes \frac{-1}{2}\,|\downarrow\rangle = -\,|\downarrow\downarrow\rangle\,.\end{aligned}$$

Hence the tensor product space contains eigenstates of τ_V^3 relative to the eigenvalues $1, 0, 0, -1$. The representations we have started with were by construction two irreducible representation of the Algebra of $SU(2)$, while in general their tensor product is not an irreducible representation. However it is always possible to decompose it in direct sum of irreducible representations $D^{1\otimes 2} \equiv D^1 \otimes D^2 = D_a \oplus D_b$.
Let's now construct explicitly these two representations. In order to do so, one first considers the basis $\mathcal{B}_V$ and takes its element with the largest eigenvalue of τ_V^3, in this case $|\uparrow\uparrow\rangle$; this state is called the *highest weight* state in the tensor product representation. The action of the raising operator on this state is

$$\tau_V^+\,|\uparrow\uparrow\rangle = \tau^+\,|\uparrow\rangle \otimes 1\,|\uparrow\rangle + 1\,|\uparrow\rangle \otimes \tau^+\,|\uparrow\rangle = 0.$$

Since $\tau_V^3\,|\uparrow\uparrow\rangle \equiv M\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$, one can write this state in notation $|J, M\rangle$ as

$$|\uparrow\uparrow\rangle \equiv |J = 1, M = 1\rangle\,,$$

where the fact that for this state $J = M$ is due to the definition of highest weight state (remember Set5, where the j labeling an irreducible representation was a shorthand notation for m_{max}). Thus we have just noticed that in the tensor product of two representations $j = \frac{1}{2}$ of $SU(2)$ there is a representation $J = 1$. To build the remaining part of the basis of this representation it is sufficient to apply $2J = 2$ times the lowering operator τ_V^-, and use the explicit knowledge about the action of τ^- on the states of the representations $j = \frac{1}{2}$.

$$\begin{aligned}\tau_V^-\,|\uparrow\uparrow\rangle &= \tau^-\,|\uparrow\rangle \otimes 1\,|\uparrow\rangle + 1\,|\uparrow\rangle \otimes \tau^-\,|\uparrow\rangle\\ &= \frac{1}{\sqrt{2}}\sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)}\,|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes \frac{1}{\sqrt{2}}\sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)}\,|\downarrow\rangle\\ &= \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\\ \equiv \tau_V^-\,|1,1\rangle &= \frac{1}{\sqrt{2}}\sqrt{1\,(1+1) - 1\,(1-1)}\,|1,0\rangle = |1,0\rangle\,,\end{aligned}$$

$$\begin{aligned}\tau_V^-\,\frac{1}{\sqrt{2}}\,(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) &= \frac{1}{\sqrt{2}}(\tau^-\,|\uparrow\rangle \otimes 1\,|\downarrow\rangle + 1\,|\downarrow\rangle \otimes \tau^-\,|\uparrow\rangle)\\ &= \frac{1}{2}\sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)}\,|\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes \frac{1}{2}\sqrt{1/2\,(1/2+1) - 1/2\,(1/2-1)}\,|\downarrow\rangle\\ &= |\downarrow\downarrow\rangle\\ \equiv \tau_V^-\,|1,0\rangle &= \frac{1}{\sqrt{2}}\sqrt{1\,(1+1) - 0\,(0-1)}\,|1,-1\rangle = |1,-1\rangle\,,\\ \tau_V^-\,|\downarrow\downarrow\rangle = 0.\end{aligned}$$

Notice that the three elements in this basis, namely $|\uparrow\uparrow\rangle \equiv |1,1\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \equiv |1,0\rangle$ and $|\downarrow\downarrow\rangle \equiv |1,-1\rangle$, have the same symmetry properties under permutations of the two spins, i.e. the raising and lowering operators don't change the symmetry properties of the states they act on; the representation $J = 1$ of SU(2) is a symmetric representation.
Note that this is a general statement: the highest weight representation (i.e. the representation containing the highest weight state) in the decomposition of a tensor product of n identical representations of $SU(2)$ is always symmetric under permutations of the particles of the component representations. This is so because the highest weight state is always of the form $|j_a, j_a\rangle \otimes |j_a, j_a\rangle \otimes ... \otimes |j_a, j_a\rangle$ and the raising/lowering operators don't modify the symmetry of the states.
Since the vector space on which the representation $\frac{1}{2} \otimes \frac{1}{2}$ acts is 4-dimensional, and we have found that one of the irreducible representations in which it decomposes is 3-dimensional, then only another irreducible 1-dimensional representation of $SU(2)$ can appear in the direct sum, and this is in fact the representation with $J = 0$ (and consequently $M = 0$). A basis for this representation is build by considering the state with $M = 0$ in the representation with $J = 1$ and finding a linear combination of the states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ (the ones with $M = 0$) orthogonal to $|1,0\rangle$:

$$\begin{aligned}0 &= \Big(\langle\downarrow| \otimes \langle\uparrow| + \langle\uparrow| \otimes \langle\downarrow|\Big)\Big(A\,|\uparrow\rangle \otimes |\downarrow\rangle + B\,|\downarrow\rangle \otimes |\uparrow\rangle\Big) \\ &= A\,\underbrace{\langle\downarrow|\uparrow\rangle\langle\uparrow|\downarrow\rangle}_{=0} + A\,\langle\uparrow|\uparrow\rangle\langle\downarrow|\downarrow\rangle + B\,\langle\downarrow|\downarrow\rangle\langle\uparrow|\uparrow\rangle + B\,\underbrace{\langle\uparrow|\downarrow\rangle\langle\downarrow|\uparrow\rangle}_{=0} \\ &= A + B.\end{aligned}$$

Therefore the state belonging to the $J = 0$ representation is the antisymmetric combination $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$, and a prefactor of $\frac{1}{\sqrt{2}}$ ensures its correct normalization: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \equiv |0,0\rangle$.
The advantage of performing such a decomposition is that now it is simple to write the action of the algebra on this vector space: organizing the basis as follows

$$\mathcal{B}_V = \left\{ |\uparrow\uparrow\rangle\,,\ \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}},\ |\downarrow\downarrow\rangle\,;\ \ \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right\},$$

and calling

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

any vector $v \in V$ will be written as

$$v = \begin{pmatrix} v_{J=1} \\ \hline v_{J=0} \end{pmatrix},$$

where $v_{J=1}$ is a three dimensional vector while $v_{J=0}$ is one dimensional. Moreover the generators will have the simple form

$$\tau^i_V = \left(\begin{array}{ccc|c} & & & 0 \\ & \tau^i_{J=1} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \tau^i_{J=0} \end{array}\right),$$

and the same will be for the representative of the group elements, so the representation matrices are in block-diagonal form. This proves that the tensor product representation $\frac{1}{2} \otimes \frac{1}{2}$ is fully decomposed into the direct sum $1 \oplus 0$, and thus the vector space V is as well decomposed into a direct sum of a 3-dimensional and a 1-dimensional invariant subspaces, $V = V_{J=1} \oplus V_{J=0}$, spanned respectively by the first three and by the fourth element of $\mathcal{B}_V$.

Let's summarize the steps to be followed in order to decompose a tensor product representation.

1) Build a basis for the tensor product space with all the possible combinations of vectors of the bases of the 'component' spaces (the spaces on which the 'component' representations act).

2) Find the highest weight state in the basis: this is always possible because the action of τ^3_V on the tensor product space is known in terms of the action of the τ^3 on the 'component' spaces. The representation containing the highest weight state (whose eigenvalue of τ^3_V, called *weight*, is M) has $J = M$.

3) Build the representation J by acting $2J$ times with the lowering operator τ_V^- on the highest weight state.

4) Set aside the subspace associated to the spin $J = M$ representation. Build, with the states in the basis of the tensor product space, the combinations orthogonal to the basis of the spin $J = M$ representation. Find among these combinations the state with weight $M - 1$: this is the highest weight state of the representation $J = M - 1$.

5) Reiterate the procedure from point 2) until all the states are assigned to irreducible representations.

Strongly recommended reading: *H. Georgi, Lie Algebras in Particle Physics*, chapter 3.