Quantum Field Theory

Set 4: solutions

Exercise 1

Let us write with no loss of generality:

$$D = \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right), \qquad A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right).$$

Then $[D, A] = 0$, gives the constraints:

$$\begin{cases} [D_1, A_{11}] = 0 \\ [D_2, A_{22}] = 0, \end{cases} \qquad \begin{cases} D_1 A_{12} - A_{12} D_2 = 0 \\ D_2 A_{21} - A_{21} D_1 = 0. \end{cases}$$

Then first Schur lemma implies $A_{12} = 0,\;\; A_{21} = 0$, while second Schur lemma gives $A_{11} = \lambda_1 \mathbb{1}$ and $A_{22} = \lambda_2 \mathbb{1}$.

Exercise 2

Given an algebra

$$\left[T^a, T^b\right] = i f^{abc} T^c \ ,$$

one can consider the following identity

$$\begin{aligned} &\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = \\ &T^a \left(T^b T^c - T^c T^b\right) - \left(T^b T^c - T^c T^b\right) T^a + T^b \left(T^c T^a - T^a T^c\right) - \left(T^c T^a - T^a T^c\right) T^b \\ &+ T^c \left(T^a T^b - T^b T^a\right) - \left(T^a T^b - T^b T^a\right) T^c = 0. \end{aligned}$$

Substituting in the first line the result of each commutator one gets

$$\begin{aligned} &\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] \\ &= \sum_d i f^{bcd} \left[T^a, T^d\right] + i f^{cad} \left[T^b, T^d\right] + i f^{abd} \left[T^c, T^d\right] \\ &= \sum_{d,f} - f^{bcd} f^{adf} T^f - f^{cad} f^{bdf} T^f - f^{abd} f^{cdf} T^f. \end{aligned}$$

The latter is a vanishing linear combination of generators that are a basis of the algebra, therefore the whole coefficient has to be zero:

$$\sum_d \left(f^{adf} f^{bcd} + f^{bdf} f^{cad} + f^{cdf} f^{abd}\right) = 0.$$

This identity can also be used to show that the quantities f^{abc}, called *structure constants*, provide themselves a representation of the group. Let's define a set of matrices $\{A^a\}$ as

$$\left(A^a\right)_b{}^c \equiv -i f^{abc}.$$

Then the Jacobi identity can be rewritten as

$$\begin{aligned} &f^{adf} f^{bcd} - f^{bdf} f^{acd} + f^{cdf} f^{abd} = 0, \\ &(A^b)_c{}^d (A^a)_d{}^f - (A^a)_c{}^d (A^b)_d{}^f = -i f^{abd} (A^d)_c{}^f, \\ &\left[A^b, A^a\right] = i f^{bad} A^d. \end{aligned}$$

Thus the matrices satisfy the algebra and therefore provide a representation of the group. The vector space on which these matrices act is the algebra itself. This is called *adjoint representation.*

Exercise 3

Given the group properties we know that $g(p(\alpha,\beta)) \in \mathcal{G}$, so it is a function

$$p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

Given the assumption that $\mathcal{G}$ is a Lie group we know that this function is smooth and thus it can be expanded in series in a finite neighborhood of the origin. Let's start by using the property of existence of the identity element e. In formulas we have

$$g(\alpha) = g(\alpha) \cdot e = g(\alpha) \cdot g(0) = g(p(\alpha, 0))$$

and

$$g(\alpha) = e \cdot g(\alpha) = g(0) \cdot g(\alpha) = g(p(0, \alpha))$$

This means that when one of the two argument is zero we have

$$p^i(\alpha, 0) = p^i(0, \alpha) = \alpha^i$$

where we have introduced the index i of $\mathbb{R}^n$. The most general series expansion for p^i around the origin is then

$$p^i(\alpha,\beta) = \alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b + B^i_{abc}\alpha^a\alpha^b\beta^c + D^i_{abc}\alpha^a\beta^b\beta^c + O((\alpha,\beta)^4)$$

where B^i_{abc} is symmetric in the indices a and b while D^i_{abc} is symmetric in b and c. Note that terms like α^2 are excluded by the condition we found earlier from the property of the identity element. For what follows we will only need the expansion up to the second order, so we will neglect the tensors B and D.

The tensor T^i_{ab} has in principle no symmetry properties in the two indices a and b, but we will see now that the symmetric part can always be eliminated by a change of coordinates, while the antisymmetric part can't. Note that since the choice of coordinates in a manifold is arbitrary, this means that the symmetric part of T cannot contain any information about the group structure. In this new coordinate system the product will then look like

$$p'^i(\alpha', \beta') = \alpha'^i + \beta'^i + T^i_{[ab]}\alpha'^a\beta'^b + O((\alpha,\beta)^3,$$

where we have defined the antisymmetric part of T^i_{ab} as

$$T^i_{[ab]} \equiv \frac{T^i_{ab} - T^i_{ba}}{2} = -T^i_{[ba]}.$$

A generic change of coordinates expanded up to the quadratic order in the expansion has the form

$$\alpha'^i = \alpha^i + \delta^i_{ab}\alpha^a\alpha^b + O(\alpha^3).$$

Note that δ^i_{ab} is symmetric in a and b by definition. It's then natural to try to find a particular form for δ for which the symmetric part of T cancels out. Since α, β and p are all coordinates in the manifold, they will all transform in the same way. Namely

$$\beta'^i = \beta^i + \delta^i_{ab}\beta^a\beta^b + O(\beta^3).$$

and

$$p'^i(\alpha', \beta') = p^i(\alpha,\beta) + \delta^i_{ab}p^a(\alpha,\beta)p^b(\alpha,\beta) + O(p(\alpha,\beta)^3).$$

Let's now expand the right hand side of this equation and express it in terms of α' and β'. First of all we need to invert the change of variable to express α as a function of α'. We can do this iteratively since we are working at the quadratic order in the expansion

$$\alpha^i = \alpha'^i - \delta^i_{ab}\alpha^a(\alpha')\alpha^b(\alpha') + O(\alpha'^3) = \alpha'^i - \delta^i_{ab}\alpha'^a\alpha'^b + O(\alpha'^3),$$

where in the last equality we have substituted α using the first equality. Using this result, the product function in the new coordinates then looks like

$$\begin{aligned}
p'^i(\alpha',\beta') =& p^i(\alpha,\beta) + \delta^i_{ab}p^a(\alpha,\beta)p^b(\alpha,\beta) + O(p(\alpha,\beta)^3) \\
=& \alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b + \delta^i_{ab}(\alpha^a + \beta^a)(\alpha^b + \beta^b) + O((\alpha,\beta)^3) \\
=& (\alpha'^i - \delta^i_{ab}\alpha'^a\alpha'^b) + (\beta'^i - \delta^i_{ab}\beta'^a\beta'^b) + T^i_{ab}\alpha'^a\beta'^b + \delta^i_{ab}(\alpha^a + \beta'^a)(\alpha'^b + \beta'^b) + O((\alpha',\beta')^3) \\
=& \alpha'^i + \beta'^i + T^i_{ab}\alpha'^a\beta'^b + \delta^i_{ab}(\alpha^a\beta'^b + \alpha'^b\beta'^a) + O((\alpha',\beta')^3)
\end{aligned}$$

Decomposing T into symmetric and antisymmetric pieces,

$$T^i_{ab} = T^i_{(ab)} + T^i_{[ab]},$$

where we have defined the symmetric part of T as

$$T^i_{(ab)} \equiv \frac{T^i_{ab} + T^i_{ba}}{2} = T^i_{(ba)},$$

we can write p' as

$$p'^i(\alpha', \beta') = \alpha'^i + \beta'^i + T^i_{[ab]}\alpha'^a\beta'^b + \left(\frac{T^i_{(ab)}}{2} + \delta^i_{ab}\right)(\alpha'^a\beta'^b + \alpha'^b\beta'^a) + O((\alpha', \beta')^3).$$

It is thus clear that if we choose

$$\delta^i_{ab} = -\frac{T^i_{(ab)}}{2},$$

the last piece vanishes and p'^i depends only on the antisymmetric part $T^i_{[ab]}$. From now on we will take the tensor T to be antisymmetric in a and b without loss of generality.

Let's now compute the coordinates of the inverse element

$$g(\bar{\alpha}) \equiv g(\alpha)^{-1}.$$

From the definition of inverse we have

$$g(p(\bar{\alpha}, \alpha)) = g(\bar{\alpha})g(\alpha) = e = g(0).$$

So we have the equation

$$0 = p^i(\bar{\alpha}, \alpha) = \bar{\alpha}^i + \alpha^i + T^i_{ab}\bar{\alpha}^a\alpha^b + o((\alpha, \beta)^3).$$

We can solve this equation iteratively for $\bar{\alpha}$ as a function of α to find

$$\bar{\alpha}^i = -\alpha^i + T^i_{ab}\alpha^a\alpha^b + o((\alpha, \beta)^3) = -\alpha^i + o((\alpha, \beta)^3)$$

where the last equality is because T is antisymmetric.

Let's now compute the commutator

$$g(c(\alpha, \beta)) \equiv g^{-1}(\alpha)g^{-1}(\beta)g(\alpha)g(\beta)$$

for α and β close to the origin. Let's rewrite the product of the first two group elements as such

$$g^{-1}(\alpha)g^{-1}(\beta) = (g(\beta)g(\alpha))^{-1} = g(\bar{p}(\beta, \alpha)).$$

In this way the commutator is given by

$$c^i(\alpha, \beta) = p^i(\bar{p}(\beta, \alpha), p(\alpha, \beta))).$$

Now we just have to expand this product using the formulas we found previously (for sake of notation, we will omit the $o((\alpha, \beta)^3)$ in every equality)

$$\begin{aligned}
c^i(\alpha, \beta) &= p^i(\bar{p}(\beta, \alpha), p(\alpha, \beta)) \\
&= \bar{p}^i(\beta, \alpha) + p^i(\alpha, \beta) + T^i_{ab}\bar{p}^a(\beta, \alpha)p^b(\alpha, \beta) \\
&= -p^i(\beta, \alpha) + p(\alpha, \beta) + T^i_{ab}(-p^a(\beta, \alpha))p^b(\alpha, \beta) \\
&= -(\beta^i + \alpha^i + T^i_{ab}\beta^a\alpha^b) + (\alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b) + T^i_{ab}(-(\beta^a + \alpha^a))(\alpha^b + \beta^b) \\
&= (T^i_{ab} - T^i_{ba})\beta^a\alpha^b
\end{aligned}$$

where in the last line we have again dropped some terms due to the antisymmetry of T. The computation we just made proves that $c(\alpha, \beta)$ close to the identity is linear in its arguments and antisymmetric.

The last thing we need to show to identify c with the Lie product is that it satisfies the Jacobi identity. We will now see that this comes from the associative property of the group

$$p^i(\alpha, p(\beta,\gamma)) = p^i(p(\alpha,\beta),\gamma)$$

Since the Jacobi identity involves product of two T symbols, we will need to expand p to the cubic order, that is restore the tensors B and D previously introduced. Let's expand the left-hand side of this equation

$$\begin{aligned}
p^i(\alpha, p(\beta,\gamma)) =& \alpha^i + p^i(\beta,\gamma) + T^i_{ab}\alpha^a p^b(\beta,\gamma) + B^i_{abc}\alpha^a\alpha^b p^c(\beta,\gamma) + D^i_{abc}\alpha^a p^b(\beta,\gamma)p^c(\beta,\gamma)\\
=& \alpha^i + (\beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + B^i_{abc}\beta^a\beta^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c)\\
&+ T^i_{ab}\alpha^a(\beta^b + \gamma^b + T^b_{cd}\beta^c\gamma^d)\\
&+ B^i_{abc}\alpha^a\alpha^b(\beta^c+\gamma^c) + D^i_{abc}\alpha^a(\beta^b+\gamma^b)(\beta^c+\gamma^c)\\
=& \alpha^i + \beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + T^i_{ab}\alpha^a\beta^b + T^i_{ab}\alpha^a\gamma^b + T^i_{ab}T^b_{cd}\alpha^a\beta^c\gamma^d\\
&+ B^i_{abc}\beta^a\beta^b\gamma^c + B^i_{abc}\alpha^a\alpha^b\beta^c + B^i_{abc}\alpha^a\alpha^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c + D^i_{abc}\alpha^a\beta^b\beta^c + D^i_{abc}\alpha^a\gamma^b\gamma^c\\
&+ 2D^i_{abc}\alpha^a\beta^b\gamma^c.
\end{aligned}$$

In the last step we have used explicitly the symmetry properties of B and D. The right hand side gives a very similar result

$$\begin{aligned}
p^i(p(\alpha,\beta),\gamma) =& \alpha^i + \beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + T^i_{ab}\alpha^a\beta^b + T^i_{ab}\alpha^a\gamma^b + T^i_{ab}T^a_{cd}\alpha^b\beta^c\gamma^d\\
&+ B^i_{abc}\beta^a\beta^b\gamma^c + B^i_{abc}\alpha^a\alpha^b\beta^c + B^i_{abc}\alpha^a\alpha^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c + D^i_{abc}\alpha^a\beta^b\beta^c + D^i_{abc}\alpha^a\gamma^b\gamma^c\\
&+ 2B^i_{abc}\alpha^a\beta^b\gamma^c.
\end{aligned}$$

Many of the terms simplify leading to the equation

$$T^i_{ab}T^b_{cd}\alpha^a\beta^c\gamma^d - T^i_{ab}T^a_{cd}\alpha^b\beta^c\gamma^d = 2B^i_{abc}\alpha^a\beta^b\gamma^c - 2D^i_{abc}\alpha^a\beta^b\gamma^c.$$

Equivalently, collecting α, β and γ

$$T^i_{ak}T^k_{bc} - T^i_{kc}T^k_{ab} = 2B^i_{abc} - 2D^i_{abc}.$$

The trick to recover the Jacobi identity is now to do an antisymmetric sum over all the permutation of the indices a, b and c, that is, calling this equation $eq(a,b,c)$, to compute

$$eq(a,b,c) + eq(b,c,a) + eq(c,a,b) - eq(b,a,c) - eq(a,c,b) - eq(c,b,a).$$

In this way, the terms on the right hand side will sum out to zero, since they are symmetric on the indices (a,b) and (b,c) respectively. The left hand also simplifies greatly by using the antisymmetry property of T, leaving only three terms

$$T^i_{ka}T^k_{bc} + T^i_{kb}T^k_{ca} + T^i_{kc}T^k_{ab} = 0$$

that is, the Jacobi identity.

Exercise 4

In this exercise we will study the finite group $\mathcal{S}_3$. This is the group of permutations of three objects. Each element of the group is a rearrangement of a given set of three objects. We can easily list all the possible rearrangements of the set $\{1,2,3\}$

$$\begin{gathered}
e : \{1,2,3\} \to \{1,2,3\},\ a_1 : \{1,2,3\} \to \{2,3,1\},\ a_2 : \{1,2,3\} \to \{3,1,2\},\\
a_3 : \{1,2,3\} \to \{2,1,3\},\ a_4 : \{1,2,3\} \to \{1,3,2\}, a_5 : \{1,2,3\} \to \{3,2,1\}.
\end{gathered}$$

The number of these rearrangements is the *order* of the group and it is usually denoted as $|\mathcal{S}_3|$. In our case this is clearly

$$|\mathcal{S}_3| = 3! = 6.$$

It is easy to convince yourself that this is a group: any combinations of permutations is still a permutation, e is the identity matrix and since every permutation is a bijection it is invertible and associativity holds. From the basic definitions we just saw we can build the product table

$\times$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_3	a_3	a_4	a_5	e	a_1	a_2
a_4	a_4	a_5	a_3	a_2	e	a_1
a_5	a_5	a_3	a_4	a_1	a_2	e

We now want to find a representation of this group, that is an explicit realization of these 6 transformations using matrices. The most obvious way to do it is to consider a basis of three vectors $\{|1\rangle, |2\rangle, |3\rangle\}$ and building matrices that implement this rearranging. For example we can look for a matrices that implements the swap between the first two basis vector and this will be a representation of the group element a_3 on a vector space of dimension 3

$$D_3[a_3]\,|1\rangle = |2\rangle\,,\ D_3[a_3]\,|2\rangle = |1\rangle\,,\ D_3[a_3]\,|3\rangle = |3\rangle\,.$$

In matrix form this just corresponds to

$$D_3[a_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can do the same for the swap between $|2\rangle$ and $|3\rangle$ to find a representation of a_4

$$D_3[a_4] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The other matrices are then built by multiplying this two.

$$D_3[a_3]D_3[a_4] = D_3[a_1] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_3[a_4]D_3[a_3] = D_3[a_2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D_3[a_1]D_3[a_3] = D_3[a_5] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Finally the identity element is obviously represented by

$$D_3[e] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can represent groups with matrices with any dimension. In this case we chose 3×3 matrices for simplicity, but we might wonder if there are smaller dimension representations. In other words we want to see if this representation is *reducible*. To see this se need to find *invariant subspace*. There is an obvious invariant subspace given by the vectors of the form

$$U = \{|u\rangle = \alpha(|1\rangle + |2\rangle + |3\rangle), \forall \alpha \in \mathbb{R}\}$$

since permuting any of the basis vectors maps $|u\rangle$ into itself. This is a 1-dimensional vector space, so there must be another invariant subspace that is 2-dimensional and orthogonal to this. A basis vector for the U space is

$$|u\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The orthogonal space to this is given by

$$V = \{|v\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle + \alpha_3 |3\rangle), \forall (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 0\}.$$

This subspace is also invariant since the condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$ is left invariant by permutations. We can pick the following two orthogonal basis vectors for this space

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad |v_1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

If we change the basis from our initial basis to this we will see that our 3×3 matrices will become block diagonal. The matrix that implements the change of basis is simply given by stacking these three basis vector as column of a matrix

$$S = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

and we can use this to go to the new basis as follows

$$D_3'[g] = S^T D_3[g] S.$$

D_3' is another 3-dimensional representation and it's equivalent to D_3, since they are related by a similarity transformation. This representation looks as follows

$$D_3'[e] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad D_3'[a_1] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{array}\right), \quad D_3'[a_2] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{array}\right),$$
$$D_3'[a_3] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad D_3'[a_4] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{array}\right), \quad D_3'[a_5] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{array}\right).$$

You can see that all the matrices are in block diagonal form, where the top-left block is always a 1 while the bottom-right is a 2×2 block. The first block is the trivial representation of $\mathcal{S}_3$ called D_0, where we represent each group element with the number 1, while the second block is a 2-dimensional representation of the group D_2.

D_0 is obviously irreducible. To show that D_2 is also irreducible requires showing that it does not possess any invariant subspace. Since it is a 2 dimensional representation the only non-trivial invariant subspace it can have is 1-dimensional. A line is spanned by a single basis vector v. It can only be invariant if acting on v with any matrix belonging to D_2 gives another vector which is proportional to v, say λv with $\lambda \neq 0$.

To be concrete, given

$$v = (\alpha_1, \alpha_2), \tag{1}$$

we want to show that for all $g \in S_3$,

$$D(g)v = \lambda v \tag{2}$$

does not admit a solution for some α_1 and α_2.

Taking for example

$$D_2(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

we find that it has the following eigenvectors, i.e. it solves eq. (2), for

$$v = (1,0), \qquad \text{or} \qquad v = (0,1). \tag{4}$$

If we now take, for example,

$$D_2(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \tag{5}$$

and test (2) on either of the v's given in (4) we find that there is no $\lambda \neq 0$ that satisfies it.

We conclude that there is no (non-trivial) invariant subspace. We therefore found the following decomposition of D_3 in irreducible representations

$$D_3 = D_0 \oplus D_2.$$

The 2×2 matrix representation looks like rotation matrices of 60° angles. This is because the group $\mathcal{S}_3$ is isomorphic to the group $\mathcal{D}_3$ that are the symmetries of the equilateral triangles (rotations + reflections).