Quantum Field Theory

Set 1: solutions

Exercise 1

The *system of natural units* is a choice of units of measure commonly employed in QFT in order to simplify the notation. It corresponds to measuring velocities in units of speed of light c and quantities with dimensions of an action in units of $\hbar$. Using this trick, one can reduce every physical unit to the only remaining one, the energy, usually measured in eV (or GeV = 10^9 eV). In the following we use the fact that $c \sim L/T$ and $\hbar \sim ML^2/T$.

1. $M \sim M\dfrac{c^2}{c^2} \sim \dfrac{\text{eV}}{c^2} \sim \text{eV}$
2. $T \sim ML^2/\hbar \sim \dfrac{c^2}{\hbar}MT^2 \implies T \sim 1/M \sim (\text{eV})^{-1}$
3. $L \sim cT \sim (\text{eV})^{-1}$
4. $v \sim \dfrac{L}{T} \sim (\text{eV})^0$
5. $F \sim \dfrac{ML}{T^2} \sim (\text{eV})^2 \dfrac{1}{\hbar c} \sim (\text{eV})^2$
6. Coulomb Force $\sim \dfrac{e^2}{L^2} \Rightarrow e^2 \sim FL^2 \sim (\text{eV})^0$
7. $E \sim \dfrac{F_e}{e} \sim (\text{eV})^2$
8. Lorentz Force $\sim evB \Rightarrow B \sim \dfrac{F_M}{ev} \sim (\text{eV})^2$

Notice that one could infer the dimensionality of time and length considering for example an incoming wave with energy E. The associated frequency (inverse of time) is defined by $\nu = E/h$ and the wavelength by $\lambda = c/\nu$. Indeed highly energetic waves correspond to X-rays while, decreasing the energy (and correspondingly the frequency), one reaches UV-rays, IR-rays radio-waves and so on.

Exercise 2

Let's start by computing the dimensions of the objects we are given. Obviously m is a mass so $[m] = M$. Since c is a speed it has units $[c] = LT^{-1}$. Finally using the familiar relation $E = \hbar\omega$ we deduce that $[\hbar] = [E]T = ML^2T^{-1}$. We can now use these relations to combine these quantities to form an object with units of length

$$\lambda \equiv \frac{\hbar}{mc}, \qquad [\lambda] = L.$$

This is the Compton wavelength (up to a factor of 2π). It corresponds to the wavelength of a photon with energy equal to the particle's rest mass energy, $E_{\text{photon}} = mc^2$. Experiments done at these energies will necessarily involve relativistic velocities. The Compton wavelength tells us the length scale below which a relativistic, or better quantum field theoretic, description becomes necessary.

For the proton ($m_p = 938\,\text{MeV}/c^2$) we get

$$\lambda = \frac{\hbar}{mc} = \frac{\hbar c}{938\,\text{MeV}} = \frac{197}{938}\,\text{fm} \approx 2.10 \times 10^{-14}\,\text{cm}.$$

Similarly for the electron ($m_e = 0.511$ MeV) we get $\lambda = 3.86 \times 10^{-11}$ cm.

The mass uncertainty has units of mass, $[\Gamma] = [M] = [E][c]^{-2}$. Thus, using $[\hbar] = [E][T]$ we can easily construct an object with units of time,

$$\tau \equiv \frac{\hbar}{\Gamma c^2}, \qquad [\Gamma] = T.$$

Since $\Gamma c^2 = \Delta E$ is an energy uncertainty, we can see τ as a time uncertainty,

$$\Delta E\, \Delta t \sim \Gamma c^2\, \tau \gtrsim \hbar.$$

A particle with no mass uncertainty cannot be located in time ($\Delta t = \infty$). This means that in principle it will exist forever and always has existed, it's a *stable* particle. On the other hand, a particle with some *width* Γ will have an associated finite time uncertainty τ, a time interval outside which we cannot locate the particle. We can thus interpret τ as the *lifetime* of the particle. (This is made more precise in the context of quantum field theory.)

The lifetime of the Z boson is given by

$$\tau = \frac{\hbar}{2.495\,\text{GeV}} \approx \frac{6.582}{2.495} \times 10^{-25}\ \text{s} = 2.638 \times 10^{-25}\ \text{s}.$$

In the non-relativistic limit c cannot be relevant. The internal dynamics of a two-body system can only depend on the reduced mass $\mu = \frac{m_p m_e}{m_p + m_e}$ and since $m_p \gg m_e$ we have $\mu \approx m_e$ meaning that we may also discard m_p. We are left with $[m_e] = [E]L^{-2}T^2$, $[\hbar] = [E]T$ and $[e^2] = [E]L$. The units for e can be deduced from the Coulomb force (previous exercise) using $[\text{Force}] = [E]L^{-1}$. Solving for L and E we find the following quantities

$$r = \frac{\hbar^2}{e^2 m_e} \sim L, \qquad \epsilon = \frac{e^4 m_e}{\hbar^2} \sim E.$$

The Gaussian system of units expresses every quantity in units of E, L, T. One way of finding the value of the elementary charge in this system is to use the value of the fine structure constant

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}.$$

This constant is dimensionless, so its value is the same in every unit system. In our case since $4\pi\epsilon_0 = 1$ we can invert the equation to find

$$e^2 \approx \frac{1}{137}\hbar c \approx \frac{197}{137}\text{MeV} \cdot \text{fm} \approx 1.438 \times 10^{-7}\text{eV} \cdot \text{cm}.$$

Then, using $\hbar = 6.582 \times 10^{-16}\,\text{eV} \cdot \text{s}$ and $m_e = 0.511 \times 10^6\,\text{eV}/c^2$ we get

$$r \approx 5.3 \times 10^{-9}\,\text{cm}, \qquad \epsilon \approx 27.1\,\text{eV}.$$

These are, respectively, the Bohr radius and (twice) the absolute value of the Hydrogen ground state energy. Note that dimensional analysis cannot fix dimensionless numerical factors, hence the discrepancy between ϵ and 13.6 eV.

Exercise 3

- Starting with

$$d\tau = \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}, \tag{1}$$

  we factor out cdt to obtain

$$d\tau = c\sqrt{1 - \frac{v^2}{c^2}}\,dt, \qquad \text{and} \qquad S = \int L\,dt \qquad \text{with} \qquad L = \alpha c\sqrt{1 - \frac{v^2}{c^2}}. \tag{2}$$

- Expanding the Lagrangian L in $\frac{v}{c}$ we get

$$L = \alpha c - \frac{\alpha}{2c}v^2 + \ldots \tag{3}$$

  Ignoring the first term (it does not contribute to the equations of motion) we see that fixing $\alpha = -mc$ yields the Newtonian Lagrangian for a free particle.

- Using $\alpha = -mc$ we may now compute the canonical momentum $\vec{p} = \frac{\partial L}{\partial \vec{v}}$ and the Hamiltonian $H = \vec{p} \cdot \vec{v} - L$ giving

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad \text{and} \qquad H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4}$$

Expressing v in terms of p we obtain the energy-momentum relation (also known as *mass-shell* condition)

$$E = H = \sqrt{m^2c^4 + p^2c^2}. \tag{5}$$

Exercise 4

We can write the Lorentz transformations in term of the rapidity parameter η (defined in terms of the velocity $\beta = \tanh \eta$) as

$$\begin{aligned} t' &= t \cosh \eta - x \sinh \eta \\ x' &= x \cosh \eta - t \sinh \eta \end{aligned}$$

In the same way,

$$\begin{aligned} t &= t' \cosh \eta + x' \sinh \eta \\ x &= x' \cosh \eta + t' \sinh \eta \end{aligned}$$

From these equations we can easily find how the derivatives transform

$$\partial_t' = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial x}{\partial t'}\partial_x = \cosh \eta \, \partial_t + \sinh \eta \, \partial_x$$

$$\partial_x' = \frac{\partial t}{\partial x'}\partial_t + \frac{\partial x}{\partial x'}\partial_x = \sinh \eta \, \partial_t + \cosh \eta \, \partial_x$$

We now want to see that if ψ satisfies the Schroedinger equation, then the transformed ψ' in a new frame also satisfies the same equation in the new frame. To see this let's show that the following quantity is equal to zero

$$\begin{aligned} &i\partial_t'\psi'(t', x') - \sqrt{-\partial_x'^2 + m^2}\psi'(x', t') \\ =&i(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)\psi(t, x) - \sqrt{-(\cosh \eta \, \partial_x + \sinh \eta \, \partial_t)^2 + m^2}\psi(x, t) \end{aligned}$$

And we now want to show that this implies that ψ' must satisfy the Schroedinger equation in the new coordinates. To see this let's do some manipulation on the operator under the square root

$$\begin{aligned} &- (\cosh \eta \, \partial_x + \sinh \eta \, \partial_t)^2 + m^2 \\ =&- \cosh^2 \eta \, \partial_x^2 - \sinh^2 \eta \, \partial_t^2 - 2 \sinh \eta \cosh \eta \, \partial_x \partial_t + m^2 \\ =&- (1 + \sinh^2 \eta) \, \partial_x^2 - (\cosh^2 \eta - 1) \, \partial_t^2 - 2 \sinh \eta \cosh \eta \, \partial_x \partial_t + m^2 \\ =&- (\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)^2 + (\partial_t^2 - \partial_x^2 + m^2) \end{aligned}$$

where we have used the identity $\cosh \eta^2 - \sinh \eta^2 = 1$. It's easy to see that the Schroedinger equation implies that

$$-\partial_t^2 \psi(t, x) = (-\partial_x^2 + m^2)\psi(t, x)$$

This means that when acting on a ψ that solves the Schroedinger equation, the operator under the square root gives (if you are not convinced, remember how the square root of an operator is defined as an infinite series)

$\sqrt{-(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)^2 + (\partial_t^2 - \partial_x^2 + m^2)}\psi(t, x) = \sqrt{-(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)^2}\psi(t, x)$ So we get

$$\begin{aligned} &i\partial_t'\psi'(t', x') - \sqrt{-\partial_x'^2 + m^2}\psi'(x', t') \\ =&i(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)\psi(t, x) - \sqrt{-(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)^2}\psi(x, t) \\ =&i(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)\psi(t, x) - i(\cosh \eta \, \partial_t + \sinh \eta \, \partial_x)\psi(t, x) = 0 \end{aligned}$$

as we wanted to show.