Quantum Field Theory

Set 12: solutions

Exercise 1

In order to solve the Dirac equation for a massive spinor $(i\not{\partial} - m)\psi(x) = 0$ it's convenient to go to momentum space taking the Fourier transform of the field $\psi(x)$:

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} u(p) e^{-ipx},$$

where $u(p)$ is a four component Dirac spinor. Hence the Dirac equation assumes the algebraic form:

$$(\not{p} - m)u(p) = 0.$$

If $u(p)$ represents a single particle with mass m and momentum p_μ, one can consider solution in the rest frame, in which the momentum assumes the simple form $\tilde{p}^\mu = (m, 0, 0, 0)$. Then the Equation of motion reads:

$$(\tilde{\not{p}} - m)u_0 = (m\gamma^0 - m)u_0 = \begin{pmatrix} -m & m \\ m & -m \end{pmatrix} u_0 = 0,$$

and a simple solution is given by:

$$u_0 = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

for any arbitrary bispinor ξ. The coefficient $\sqrt{m}$ is chosen for future convenience. Starting from this solution one can obtain the general one applying a Lorentz boost to it. Indeed given a reference frame where the momentum of the particle is $p^\mu = (E, 0, 0, p^3) = \Lambda^\mu{}_\nu \tilde{p}^\nu$, one has:

$$\Lambda_D(\tilde{\not{p}} - m)u_0 = \Lambda_D(\tilde{\not{p}} - m)\Lambda_D^{-1}\Lambda_D u_0 = (\Lambda^\mu{}_\nu \tilde{p}^\nu \gamma_\mu - m)u(p) = (\not{p} - m)u(p),$$

where we have defined the solution for a generic momentum p, $u(p)$, as the Lorentz transformed of the solution at rest: $u(p) \equiv \Lambda_D u_0$. In order to obtain the explicit solution we need therefore the explicit expression for Λ_D, namely the Dirac representation of the boost transforming $\tilde{p}^\mu \longrightarrow p^\mu$. Without loss of generality we have chosen p^μ in the third direction, so that $\Lambda^\mu{}_\nu$ is of the form:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix} \qquad \begin{matrix} E = m\cosh(\eta) \\ p^3 = m\sinh(\eta) \end{matrix} \quad \Longrightarrow \quad \tanh(\eta) = \frac{p^3}{E}.$$

Moreover, recalling the definition of $\sinh(\eta)$ and $\cosh(\eta)$, one gets

$$\cosh(\eta) = \frac{e^\eta + e^{-\eta}}{2} \qquad \sinh(\eta) = \frac{e^\eta - e^{-\eta}}{2} \quad \Longrightarrow \quad e^{\pm\eta} = \frac{E \pm p^3}{m} \quad \Longrightarrow \quad e^{\pm\frac{\eta}{2}} = \sqrt{\frac{E \pm p^3}{m}}.$$

A general Lorentz transformation on Dirac spinors has the structure:

$$\Lambda_D = \begin{pmatrix} e^{-\frac{1}{2}(i\theta^i + \eta^i)\sigma^i} & 0 \\ 0 & e^{-\frac{1}{2}(i\theta^i - \eta^i)\sigma^i} \end{pmatrix}.$$

The present case is in fact simpler since we are dealing with a pure boost, $\theta^i = 0$, in the third direction only, $\eta^i = (0, 0, \eta)$. Therefore

$$\Lambda_D = \begin{pmatrix} e^{-\eta\sigma^3/2} & 0 \\ 0 & e^{+\eta\sigma^3/2} \end{pmatrix} = \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) 1_2 - \sinh\left(\frac{\eta}{2}\right)\sigma^3 & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) 1_2 + \sinh\left(\frac{\eta}{2}\right)\sigma^3 \end{pmatrix}$$

Still, using the definition of $\cosh(\eta)$, $\sinh(\eta)$, and $\sigma^3 = \mathrm{diag}(1,-1)$, one gets:

$$\begin{aligned}
\Lambda_D u_0 &= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right) & 0 \\ 0 & e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right) \end{pmatrix} \begin{pmatrix} \sqrt{m}\xi \\ \sqrt{m}\xi \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right) & 0 \\ 0 & \sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right) \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{E-p^3} & 0 & 0 & 0 \\ 0 & \sqrt{E+p^3} & 0 & 0 \\ 0 & 0 & \sqrt{E+p^3} & 0 \\ 0 & 0 & 0 & \sqrt{E-p^3} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&= \begin{pmatrix} E-p^3 & 0 & 0 & 0 \\ 0 & E+p^3 & 0 & 0 \\ 0 & 0 & E+p^3 & 0 \\ 0 & 0 & 0 & E-p^3 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{E-p^3\sigma^3} & 0 \\ 0 & \sqrt{E+p^3\sigma^3} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}.
\end{aligned}$$

Notice that since the matrices involved are diagonal there are no problems in defining the square root of them. One could extend the previous result for a general boost:

$$u(p) = \begin{pmatrix} \sqrt{E-p^i\sigma^i} & 0 \\ 0 & \sqrt{E+p^i\sigma^i} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{p_\mu\sigma^\mu} & 0 \\ 0 & \sqrt{p_\mu\bar{\sigma}^\mu} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}.$$

One can easily verify that this expression is indeed a solution of the Dirac equation:

$$\begin{aligned}
(\not{p} - m)u(p) &= \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{pmatrix} \begin{pmatrix} \sqrt{p_\mu\sigma^\mu} & 0 \\ 0 & \sqrt{p_\mu\bar{\sigma}^\mu} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{p_\nu\sigma^\nu}\,(-m + \sqrt{p_\rho\sigma^\rho}\sqrt{p_\mu\bar{\sigma}^\mu})\,\xi \\ \sqrt{p_\nu\bar{\sigma}^\nu}\,(-m + \sqrt{p_\rho\bar{\sigma}^\rho}\sqrt{p_\mu\sigma^\mu})\,\xi \end{pmatrix} = 0,
\end{aligned}$$

where in the last equality we have used

$$\sqrt{p_\mu\sigma^\mu}\sqrt{p_\nu\bar{\sigma}^\nu} = \sqrt{p_\mu\sigma^\mu p_\nu\bar{\sigma}^\nu} = \sqrt{p_\mu p_\nu \frac{(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)}{2}} = \sqrt{p_\mu p_\nu \eta^{\mu\nu}} = \sqrt{p^2} = m.$$

The expression for $u(p)$ written in this way is compact and easy to recall but not practical for computations. We now manipulate it in order to obtain a more manageable expression. First start from the simple configuration $p^\mu = (E, 0, 0, p^3)$. In this case one has

$$\sqrt{p\cdot\sigma} \equiv \sqrt{p_\mu\sigma^\mu} = \sqrt{E - p^3\sigma^3} = \alpha + \beta\,\sigma^3$$

(both sides have to be diagonal). The coefficients can be found by equating entry by entry the two sides of previous equation, and are $\alpha = (\sqrt{E-p^3} + \sqrt{E+p^3}\,)/2$ and $\beta = (\sqrt{E-p^3} - \sqrt{E+p^3}\,)/2$. To put them in a more convenient form, one can square and simplify them using the mass-shell condition $E^2 = (p^3)^2 + m^2$, getting $\alpha^2 = (E+m)/2$ and $\beta^2 = (E-m)/2$. Since these equations are quadratic, there is still a sign uncertainty, which can be solved by taking the non-relativistic ($p^3 \to 0$, $E \to m$) and the ultra-relativistic ($p^3 \to E$, $m/E \to 0$) limits of both sides in the equation, yielding

$$\sqrt{p\cdot\sigma} = \sqrt{\frac{E+m}{2}} - \sqrt{\frac{E-m}{2}}\,\sigma^3.$$

One can rewrite it in a more convenient form as

$$\sqrt{p\cdot\sigma} = \sqrt{\frac{E+m}{2}} - \frac{p^3\sigma^3}{\sqrt{2(E+m)}}.$$

For generic momenta one thus finds

$$\sqrt{p\cdot\sigma} = \sqrt{\frac{E+m}{2}} - \frac{p^i\sigma^i}{\sqrt{2(E+m)}},$$
$$\sqrt{p\cdot\bar{\sigma}} = \sqrt{\frac{E+m}{2}} + \frac{p^i\sigma^i}{\sqrt{2(E+m)}}.$$

Finally

$$u(p) = \begin{pmatrix} \left(\sqrt{\frac{E+m}{2}} - \frac{p^i\sigma^i}{\sqrt{2(E+m)}}\right)\xi \\ \left(\sqrt{\frac{E+m}{2}} + \frac{p^i\sigma^i}{\sqrt{2(E+m)}}\right)\xi \end{pmatrix}.$$

Exercise 2

The aim of this exercise is to work out a basis of 4 × 4 matrices acting on Dirac spinors which have definite transformation properties under the Lorentz group in the following sense: call Γ such a matrix and consider a basis $\mathcal{B} = \{\Gamma_1, \dots \Gamma_{16}\}$; hence Γ can be decomposed in a linear combination: $\Gamma = \sum_n e_n \Gamma_n$, with complex coefficients e_n. For a given Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ one can take the Dirac representation Λ_D and consider:

$$\Lambda_D^{\dagger}\Gamma\Lambda_D = \Gamma' \equiv \sum_n e'_m \Gamma_m.$$

We are looking for a basis such that the coefficients e_m and e'_n are related by a matrix which is some representation of $\Lambda^{\mu}{}_{\nu}$ (a sum if irreducible representation, actually).

Note. Saying that Γ has definite transformation properties under Lorentz has to be further specified. Actually, the Γ's are all numbers under the Lorentz group, since Lorentz transformations act on the Hilbert space of physical states, while the Γ's are matrices on a 4-dimensional complex space (the one in which Dirac indices live) which is a *different* space: on the Hilbert space, the Dirac matrices are just numerical coefficients that stay unchanged under any physical transformation. Nevertheless, given a Dirac spinor ψ, the object $O = \bar{\psi}\Gamma\psi$ transforms under Lorentz into $\bar{\psi}\Lambda_D^{-1}\Gamma\Lambda_D\psi$, so it is useful to have a basis of Γ's such that $\Lambda_D^{-1}\Gamma\Lambda_D = \Lambda|_{rep}\Gamma$, for $\Lambda|_{rep}$ some Lorentz representation matrix, because in this way O can be used to build lagrangian densities involving Dirac fermions. Notice that this is analogous to when one takes the object $O' = \Psi_i^{\dagger}\sigma^a_{ij}\Psi_j$ and performs an Isospin transformation on the fields: the σ's do not transform, since the Isospin transformation is again a transformation on the Hilbert space of physical states, and the Pauli matrices are just numerical coefficients in that space. So in what follows, when referring to Γ with definite transformation properties under the Lorentz group, it will always be understood the reference to $\bar{\psi}\Gamma\psi$.

The space of 4 × 4 matrices has complex dimension 16, meaning that its basis is formed by 16 elements each independent with respect to *complex* linear combinations of the others. Clearly we can take the identity which is a scalar under the transformation defined. Secondly one can consider the four Dirac matrices γ^{μ}, which one knows transform as a four vector (see Set 12, exercise 1, for the explicit proof):

$$\Lambda_D^{-1}\gamma^{\mu}\Lambda_D = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}.$$

Starting from last relation one can easily obtain other objects transforming according to irreducible representation of the Lorentz group: $\gamma^{\mu\nu} \equiv \gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2!}[\gamma^{\mu},\gamma^{\nu}]$, $\gamma^{\mu\nu\rho} \equiv \frac{1}{3!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}$, $\gamma^{\mu\nu\rho\sigma} \equiv \frac{1}{4!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}$, where we have used the notation [...] among the indices to mean their complete antisymmetrization. In order to extract the transformation properties of these quantities one has to generalize the information that γ^{μ} is a Lorentz four vector:

$$\Lambda_D^{-1}\gamma^{\mu\nu}\Lambda_D = \Lambda_D^{-1}\frac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\Lambda_D = \frac{1}{2}\Lambda_D^{-1}(\gamma^{\mu}\Lambda_D\Lambda_D^{-1}\gamma^{\nu} - \gamma^{\nu}\Lambda_D\Lambda_D^{-1}\gamma^{\mu})\Lambda_D = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\gamma^{\rho\sigma},$$
$$\Lambda_D^{-1}\gamma^{\mu\nu\alpha}\Lambda_D = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\Lambda^{\alpha}{}_{\beta}\gamma^{\rho\sigma\beta},$$
$$\Lambda_D^{-1}\gamma^{\mu\nu\alpha\delta}\Lambda_D = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\Lambda^{\alpha}{}_{\beta}\Lambda^{\delta}{}_{\gamma}\gamma^{\rho\sigma\beta\gamma}.$$

Therefore these tensors transform respectively as antisymmetric tensors of rank $r = 2$, 3 and 4, respectively, and each of them contains $\frac{4!}{(4-r)!r!}$ independent matrices (recall that these objects are Lorentz tensors since they carry spacetime indices and each component is a 4×4 matrix acting on Dirac indices). In the following table we summarize all the matrices:

Basis element	Trasformation property	Components
1	Scalar	1
γ^μ	Four vector	4
$\gamma^{\mu\nu}$	Antisymmetric rank 2	6
$\gamma^{\mu\nu\rho}$	Antisymmetric rank 3	4
$\gamma^{\mu\nu\rho\sigma}$	Antisymmetric rank 4	1

The tensors so defined are a total of 16 independent matrices, and therefore any matrix acting on Dirac indices can be expressed as a linear combination of them with complex coefficients. This basis be re-expressed in a more compact form introducing an additional 4×4 matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$
$$\gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0$$

In writing γ^5 in terms of the Levi-Civita tensor we have used the fact that every exchange of two *different* γ's implies a minus sign, and the fact that $\varepsilon_{0123} = -\varepsilon^{0123} = -1$ to determine the overall sign. Note that the definition of this matrix involves the antisimmetrization of four gamma matrices, which is very similar to the rank 4 tensor that we had before. One can indeed show the following relations:

$$\gamma^{\mu\nu\rho\sigma} = -i\gamma^5\varepsilon^{\mu\nu\rho\sigma},$$
$$\gamma^{\mu\nu\rho} = -i\gamma^5\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma.$$

The former relation can be immediately proved considering that in four dimensions there is only one independent completely antisymmetric tensor with four indices, namely the Levi-Civita tensor, so every object with those characteristics has to be proportional to it, and the proportionality coefficient can be worked out considering the 0123 component. The second relation can be worked out in components in the following way: $\gamma^{\mu\nu\rho\sigma}\gamma_\sigma = \gamma^0\gamma^1\gamma^2\gamma^3\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma = \varepsilon^{\mu\nu\rho 3}\gamma^0\gamma^1\gamma^2 - \varepsilon^{\mu\nu\rho 2}\gamma^0\gamma^1\gamma^3 + \varepsilon^{\mu\nu\rho 1}\gamma^0\gamma^2\gamma^3 - \varepsilon^{\mu\nu\rho 0}\gamma^1\gamma^2\gamma^3$, so that

$$\gamma^{012\sigma}\gamma_\sigma = \gamma^0\gamma^1\gamma^2, \qquad \gamma^{013\sigma}\gamma_\sigma = \gamma^0\gamma^1\gamma^3,$$
$$\gamma^{023\sigma}\gamma_\sigma = \gamma^0\gamma^2\gamma^3, \qquad \gamma^{123\sigma}\gamma_\sigma = \gamma^1\gamma^2\gamma^3,$$

and it is 0 in all other cases. Likewise,

$$\gamma^{012} = \gamma^0\gamma^1\gamma^2, \qquad \gamma^{013} = \gamma^0\gamma^1\gamma^3,$$
$$\gamma^{023} = \gamma^0\gamma^2\gamma^3, \qquad \gamma^{123} = \gamma^1\gamma^2\gamma^3,$$

so $\gamma^{\mu\nu\rho\sigma}\gamma_\sigma = \gamma^{\mu\nu\rho}$, hence the relation we wanted.
Using these relations one can identify the 16 independent matrices in:

$$\mathcal{B} = \left\{1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \gamma^{\mu\nu}\right\}.$$

Now we know how to construct all the possible spinor bilinear in the Lagrangian, and they transformation properties under Lorentz transformation.

Exercise 3

Massive spinor and anti-spinor are given by, respectively,

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} & 0 \\ 0 & \sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}, \qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} & 0 \\ 0 & -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}.$$

Given that:

$$\gamma^0\gamma^\mu = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}$$

we have:

$$\bar{u}^r(\vec{p})\gamma^\mu u^s(\vec{p}) = \bar{v}^r(\vec{p})\gamma^\mu v^s(\vec{p}) = (\xi^r)^\dagger \left[\sqrt{p\cdot\sigma}\bar{\sigma}^\mu\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}\sigma^\mu\sqrt{p\cdot\bar{\sigma}}\right]\xi^s$$

Let us calculate the first term:

$$\sqrt{p_\mu\sigma^\mu}\bar{\sigma}^0\sqrt{p\cdot\sigma} = \sqrt{p\cdot\sigma}\mathbb{1}\sqrt{p\cdot\sigma} = p\cdot\sigma\,,$$

$$\begin{aligned}\sqrt{p\cdot\sigma}\bar{\sigma}^i\sqrt{p\cdot\sigma} &= -\sqrt{p\cdot\sigma}\sigma^i\sqrt{p\cdot\sigma} = -\frac{1}{2}\left(\sqrt{E+m}-\frac{\vec{p}\cdot\vec{\sigma}}{\sqrt{E+m}}\right)\sigma^i\left(\sqrt{E+m}-\frac{\vec{p}\cdot\vec{\sigma}}{\sqrt{E+m}}\right)\\ &= -\frac{1}{2}\left((E+m)\sigma^i - \left\{\vec{p}\cdot\vec{\sigma},\sigma^i\right\} + \frac{(\vec{p}\cdot\vec{\sigma})\sigma^i(\vec{p}\cdot\vec{\sigma})}{E+m}\right) = \frac{1}{2}\left[(E+m)\sigma^i - 2p^i + \frac{p^2\sigma^i + 2p^i(p\cdot\sigma)}{E+m}\right]\end{aligned}$$

where we made use of the anti-commutation relation $\left\{\sigma^i,\sigma^j\right\} = 2\delta^{ij}\mathbb{1}$. After calculating the second term in the same way, the final result is

$$\sqrt{p\cdot\sigma}\bar{\sigma}^0\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}\sigma^0\sqrt{p\cdot\bar{\sigma}} = p\cdot\bar{\sigma} + p\cdot\bar{\sigma} = 2p^0\mathbb{1}_2\,,$$

$$\sqrt{p\cdot\sigma}\bar{\sigma}^i\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}\sigma^i\sqrt{p\cdot\bar{\sigma}} = \frac{1}{2}(2p^i + 2p^i)\mathbb{1} = 2p^i\mathbb{1}_2\,,$$

which implies

$$\bar{u}^r(\vec{p})\gamma^\mu u^s(\vec{p}) = \bar{v}^r(\vec{p})\gamma^\mu v^s(\vec{p}) = 2p^\mu\delta^{rs}\,.$$

Using $(\gamma^0)^2 = \mathbb{1}$, this results immediately implies that the scalar product of two spinors is not Lorentz invariant:

$$u^{r\,\dagger}(\vec{p})u^s(\vec{p}) = v^{r\,\dagger}(\vec{p})v^s(\vec{p}) = 2p^0\delta^{rs}\,.$$

In order to define a Lorentz invariant product, we consider instead the scalar product between $u^r(\vec{p})$ $(v^r(\vec{p}))$ and $\bar{u}^r(\vec{p})$ $(\bar{v}^r(\vec{p}))$. To compute the latter, recall:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0\end{pmatrix}\,,$$

from which we find

$$\gamma_0 u^s(\vec{p}) = \begin{pmatrix} 0 & \sqrt{p\cdot\bar{\sigma}} \\ \sqrt{p\cdot\sigma} & 0\end{pmatrix}\begin{pmatrix}\xi^s \\ \xi^s\end{pmatrix} = \begin{pmatrix}\sqrt{p\cdot\bar{\sigma}}\xi^s \\ \sqrt{p\cdot\sigma}\xi^s\end{pmatrix}\,, \qquad \gamma_0 v^s(\vec{p}) = \begin{pmatrix} 0 & -\sqrt{p\cdot\bar{\sigma}} \\ \sqrt{p\cdot\sigma} & 0\end{pmatrix}\begin{pmatrix}\xi^s \\ \xi^s\end{pmatrix} = \begin{pmatrix}-\sqrt{p\cdot\bar{\sigma}}\xi^s \\ \sqrt{p\cdot\sigma}\xi^s\end{pmatrix}\,.$$

It follows that

$$\bar{u}^s(\vec{p})u^r(\vec{p}) = -\bar{v}^s(\vec{p})v^r(\vec{p}) = (\xi^s)^\dagger\left[\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} + \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\right]\xi^r = 2m(\xi^s)^\dagger\xi^r = 2m\delta^{rs}\,,$$

where we used the following equation proved in exercise 1: $\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} = \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} = m$. Similarly, we find

$$\bar{u}^s(\vec{p})v^r(\vec{p}) = -\bar{v}^s(\vec{p})u^r(\vec{p}) = (\xi^s)^\dagger\left[\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} - \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\right]\xi^r = 0\,.$$

Notice also that from the definition of σ^μ and $\bar{\sigma}^\mu$ it follows that

$$u^s(-\vec{p}) = \gamma_0 u^s(\vec{p})\,, \qquad v^s(-\vec{p}) = -\gamma_0 u^s(\vec{p})\,.$$

Then the previous result implies $u^{r\,\dagger}(\vec{p})v^s(-\vec{p}) = v^{r\,\dagger}(\vec{p})u^s(-\vec{p}) = 0$.
Finally, we can prove the following relations

$$\sum_{r=1,2} u^r(\vec{p})\bar{u}^r(\vec{p}) = \not{p} + m\,, \qquad \sum_{r=1,2} v^r(\vec{p})\bar{v}^r(\vec{p}) = \not{p} - m\,.$$

To this aim, we observe that

$$\sum_{s=1,2}\xi_s\xi_s^\dagger = \mathbb{1}_2\,.$$

Then

$$\begin{aligned}\sum_{r=1,2} u^r(\vec{p})\bar{u}^r(\vec{p}) &= \begin{pmatrix}\sqrt{p\cdot\sigma} & 0 \\ 0 & \sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2\end{pmatrix}\begin{pmatrix}0 & \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} & 0\end{pmatrix} = \begin{pmatrix}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\begin{pmatrix}0 & \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} & 0\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m\end{pmatrix}\\ &= \not{p} + m\,,\end{aligned}$$

where we used again $\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} = m$ and the definition of gamma matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} .$$

Similarly, we find

$$\begin{aligned}
\sum_{r=1,2} v^r(\vec{p})\bar{v}^r(\vec{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma} & 0 \\ 0 & -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar{\sigma}} & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma} & \sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar{\sigma}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} \\
&= \not{p} - m \,.
\end{aligned}$$