Quantum Field Theory

Set 11: solutions

Exercise 1

- We have

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\sigma J_{\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}, \tag{1}$$

  where in the last equality we relabeled $\sigma \leftrightarrow \nu$ and used anti-symmetry of $\epsilon^{\mu\nu\rho\sigma}$ and $J_{\rho\sigma}$.

  Now all we have to show is that the term involving the commutator is zero. It follows from the algebra (eq. (3.191) in the lecture notes),

$$[J_{\nu\rho}, P_\sigma] = i(\eta_{\sigma\rho}P_\nu - \eta_{\sigma\nu}P_\rho), \tag{2}$$

  that

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\sigma] = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(\eta_{\sigma\rho}P_\nu - \eta_{\sigma\nu}P_\rho) = 0. \tag{3}$$

  The first term is zero because $\eta_{\sigma\rho}$ is symmetric and its indices are contracted with $\epsilon^{\mu\nu\rho\sigma}$ which is anti-symmetric. The second term is zero for the same reason.

- 1. It follows from the algebra (eq. (3.192) from the lectures notes) and anti-symmetry of the Levi-Civita tensor that

$$W^\mu P_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma P_\mu = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}[P_\sigma, P_\mu] = 0. \tag{4}$$

  2. We have

$$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}[P^\mu, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([P^\mu, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[P^\mu, P_\sigma]). \tag{5}$$

  Now we make use of the algebra, in particular $[P^\mu, P_\sigma] = 0$ and eq. (2) to find

$$[P^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([P^\mu, J_{\alpha\rho}]P_\sigma = \frac{i}{2}\epsilon^{\nu\alpha\rho\sigma}(\delta^\mu_\alpha P_\rho - \delta^\mu_\rho P_\alpha)P_\sigma = 0, \tag{6}$$

  due to antisymmetry of the Levi-Civita tensor.

  3. We have

$$[J^{\mu\nu}, W^\rho] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma}[J^{\mu\nu}, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\rho\alpha\beta\sigma}([J^{\mu\nu}, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[J^{\mu\nu}, P_\sigma]). \tag{7}$$

  We again make use of the algebra,

$$[J^{\mu\nu}, J_{\alpha\beta}] = i(\delta^\nu_\alpha J^\mu{}_\beta - \delta^\mu_\alpha J^\nu{}_\beta + \delta^\mu_\beta J^\nu{}_\alpha - \delta^\nu_\beta J^\mu{}_\alpha), \qquad [J^{\mu\nu}, P_\sigma] = i(\delta^\nu_\sigma P^\mu - \delta^\mu_\sigma P^\nu), \tag{8}$$

  to get

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2}(\epsilon^{\rho\nu\beta\sigma}J^\mu{}_\beta P_\sigma + \epsilon^{\rho\alpha\nu\sigma}J_\alpha{}^\mu P_\sigma + \epsilon^{\rho\alpha\beta\nu}J_{\alpha\beta}P_\mu) - (\mu \leftrightarrow \nu). \tag{9}$$

  We now factor out $J_{\alpha\beta}P_\sigma$ to have

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2}J_{\alpha\beta}P_\sigma(\eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu}) - (\mu \leftrightarrow \nu). \tag{10}$$

  We now make use of the identity[1]

$$\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha}\epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta}\epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma}\epsilon^{\rho\alpha\beta\nu} \tag{12}$$

[1] This identity follows from expanding the determinant relation

$$\det(A)\epsilon^{\rho\alpha\beta\sigma} = A^\rho{}_{\rho'}A^\alpha{}_{\alpha'}A^\beta{}_{\beta'}A^\sigma{}_{\sigma'}\epsilon^{\rho'\alpha'\beta'\sigma'} \tag{11}$$

to linear order in $\omega^\rho{}_{\rho'}$, where $A^\rho{}_{\rho'} = \delta^\rho_{\rho'} + \omega^\rho{}_{\rho'} + O(\omega^2)$. The left-hand-side becomes proportional to the trace $\operatorname{tr}\omega = \eta^{\mu\nu}\omega_{\mu\nu}$, matching to the left-hand-side of (12). Correpondingly, the right-hand-sides will match and given that $\omega^{\mu\nu}$ is arbitrary, the identity (12) will hold.

to have

$$[J^{\mu\nu}, W^\rho] = \frac{i}{2} J_{\alpha\beta} P_\sigma (\eta^{\mu\nu}\epsilon^{\rho\alpha\beta\sigma} - \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma}) - (\mu \leftrightarrow \nu) = \tag{13}$$

$$= -\frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\mu\rho}\epsilon^{\nu\alpha\beta\sigma} + \frac{i}{2} J_{\alpha\beta} P_\sigma \eta^{\nu\rho}\epsilon^{\mu\alpha\beta\sigma} = i(\eta^{\nu\rho}W^\mu - \eta^{\mu\rho}W^\nu). \tag{14}$$

Note that replacing W^ρ with the momentum P^ρ in the above yields the correct commutation relation between $J^{\alpha\beta}$ and P^ρ.This result is a consequence of the fact that W^ρ transforms as a 4-vector under Lorentz transformations (just like P^ρ).

4. We have

$$[W^\mu, W^\nu] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}[W^\mu, J_{\alpha\beta}P_\sigma] = \frac{1}{2}\epsilon^{\nu\alpha\rho\sigma}([W^\mu, J_{\alpha\beta}]P_\sigma + J_{\alpha\beta}[W^\mu, P_\sigma]). \tag{15}$$

We now make use of the results found in points 2 and 3. We find

$$[W^\mu, W^\nu] = \frac{i}{2}\epsilon^{\nu\alpha\rho\sigma}(\delta^\mu_\alpha W_\rho - \delta^\mu_\rho W_\alpha)P_\sigma = i\epsilon^{\nu\alpha\rho\sigma}W_\rho P_\sigma. \tag{16}$$

- To show that W^2 is a Casimir we have to show that it commutes with the Poincare generators P^μ and $J^{\mu\nu}$. We have

$$[W^2, P^\mu] = W_\alpha[W^\alpha, P^\mu] + [W^\alpha, P^\mu]W_\alpha = 0 \tag{17}$$

due to the result derived in point 2.

We also have

$$[W^2, J^{\mu\nu}] = W_\alpha[W^\alpha, J^{\mu\nu}] + [W^\alpha, J^{\mu\nu}]W_\alpha = i[W^\mu, W^\nu] + i[W^\nu, W^\mu] = 0, \tag{18}$$

where we made use of the result derived in point 3.

Exercise 2

- At first, let us simply write the infinitesimal transformation for a Lorentz vector in terms of θ^k and η^k.

$$v'^\mu = \Lambda^\mu{}_\nu v^\nu \simeq (\delta^\mu{}_\nu + \omega^\mu{}_\nu)v^\nu = (\delta^\mu{}_\nu + g_{\nu\alpha}\omega^{\mu\alpha})v^\nu$$

$$v'^0 = v^0 + g_{\nu\alpha}\omega^{0\alpha}v^\nu = v^0 + g_{\nu i}\omega^{0i}v^\nu = v^0 - \delta_{ji}\omega^{0i}v^j = v^0 + \eta^i v^i \tag{19}$$

$$\begin{aligned} v'^i =& v^i + g_{\nu\alpha}\omega^{i\alpha}v^\nu = v^i + g_{\nu j}\omega^{ij}v^\nu + g_{\nu 0}\omega^{i0}v^\nu = v^i + g_{kj}\omega^{ij}v^k + g_{00}\omega^{i0}v^0 \\ =& v^i + g_{kj}\epsilon^{ijl}\theta^l v^k + g_{00}\eta^i v^0 = v^i - \epsilon^{ijl}\theta^l v^j + \eta^i v^0 \end{aligned} \tag{20}$$

where $g_{\mu\nu}$ is the Minkowski metric.

- Now, we want to show:

$$\Lambda_L(\theta,\eta)v^\mu\sigma_\mu\Lambda_L^\dagger(\theta,\eta) = \Lambda^\mu{}_\nu(\theta,\eta)v^\nu\sigma_\mu \tag{21}$$

Let us call $v \equiv v^\mu\sigma_\mu$. Expanding the left hand side to linear order:

$$\begin{aligned} v'^\mu\sigma_\mu =& \exp[-(\vec{\eta} + i\vec{\theta})\cdot\vec{\sigma}/2]\ v\ \exp[-(\vec{\eta} - i\vec{\theta})\cdot\vec{\sigma}/2] \simeq [1 - (\vec{\eta} + i\vec{\theta})\cdot\vec{\sigma}/2]\ v\ [1 - (\vec{\eta} - i\vec{\theta})\cdot\vec{\sigma}/2] \\ =& v - \frac{\eta^i}{2}\{\sigma^i, v\} - \frac{i\theta^i}{2}[\sigma^i, v] = v - \frac{\eta^i}{2}(2v^0\sigma^i - 2v^i\sigma^0) + \frac{i\theta^i}{2}(2i\epsilon^{ijk}v^j\sigma^k) \\ =& v + \eta^i v^i\sigma_0 + (+\epsilon^{ijl}\theta^l v^j + \eta^i v_0)\sigma_i. \end{aligned}$$

which, by equating the components of σ_μ on both sides of the equation, corresponds exactly to the transformations (19), (20).

- Take the complex conjugate of equation (21) and sandwich it between ϵ^{-1} and ϵ. The right-hand side becomes:

$$\Lambda^\mu{}_\nu v^\nu \epsilon^{-1}\sigma_\mu^*\epsilon = \Lambda^\mu{}_\nu v^\nu\bar{\sigma}_\mu$$

where we used one of the properties given in the text. The left-hand side becomes:

$$v^\mu\epsilon^{-1}\Lambda_L^*\sigma_\mu^*\Lambda_L^T\epsilon = v^\mu\epsilon^{-1}\Lambda_L^*\epsilon\,\epsilon^{-1}\sigma_\mu^*\epsilon\,\epsilon^{-1}\Lambda_L^T\epsilon = v^\mu\Lambda_R\bar{\sigma}_\mu\Lambda_R^\dagger$$

because $\epsilon^{-1}\Lambda_L^*\epsilon = \Lambda_R$, which also implies $\epsilon^{-1}\Lambda_L^T\epsilon = \Lambda_R^\dagger$. Therefore, we proved the equation outlined in the text.

- The last point derives from algebraic manipulations, given the last two point proven in the exercise.

The importance of this exercise lies in the fact that now we can construct spinor bilinears involving Weyl fermions and Dirac fermions, which can be used to write down interaction terms in the Lagrangian, for instance:

$$A^\mu \psi_L^\dagger \bar{\sigma}_\mu \psi_L, \quad A^\mu \psi_R^\dagger \sigma_\mu \psi_R$$

This is the way the Weyl spinors are coupled to the electromagnetic field.

Exercise 3

- Recalling the definition of $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, one can easily check:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \begin{cases} \mu = 0, \nu = 0 & \sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0 = \quad 2, \\ \mu = 0, \nu = i & \sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0 = \quad -\sigma^i + \sigma^i = 0, \\ \mu = i, \nu = j & \sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i = \quad -\sigma^i\sigma^j - \sigma^j\sigma^i = \quad -\{\sigma^i, \sigma^j\} = -2\delta^{ij}. \end{cases}$$

  In compact notation:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu}$$

  Therefore, given the definition of the Dirac matrices, we can deduce the anticommutation relation:

$$\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + (\mu \leftrightarrow \nu) = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} + (\mu \leftrightarrow \nu) = 2g^{\mu\nu}\mathbb{1}.$$

- Let us consider the second term in the right hand side of the equation:

$$\{\gamma^\mu, \gamma^5\} = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu.$$

  and let us bring γ^μ on the left of all the other matrices γ^ν, $\nu = 0, 1, 2, 3$. When $\mu \neq \nu$, the relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ gives $\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$, while when $\mu = \nu$ obviously $\gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu$. Therefore $\{\gamma^\mu, \gamma^5\} = 0$.

- The only non-trivial point of the question is to show that $(\gamma^5)^2 = 1$. This can be worked out in a way similar to the one used in the previous point.

Exercise 4

Consider the following complex scalar field doublet:

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \qquad \phi_i \in \mathbb{R}.$$

Let us write the most general Lagrangian with terms of dimension $d \leq 4$ which is invariant under $\Phi \to U\Phi$, where $U \in SU(2)$. The result is the Lagrangian given in the text:

$$\mathcal{L} = \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2. \tag{22}$$

Invariance follows from $\Phi^\dagger\Phi \to \Phi^\dagger(U^\dagger U)\Phi = \Phi^\dagger\Phi$ since $U^\dagger U = \mathbb{1}$.
To see whether eq. (22) corresponds to a *reasonable* theory, let us compute the Hamiltonian density:

$$\mathcal{H} = \partial_0\Phi^\dagger\partial_0\Phi + \partial_i\Phi^\dagger\partial_i\Phi + m^2(\Phi^\dagger\Phi) + \lambda(\Phi^\dagger\Phi)^2. \tag{23}$$

For a theory to be well defined, we need the Hamiltonian to be bounded from below. Indeed when couplings to ordinary 'healthy' matter are taken into account the system is unstable: with zero net energy one can excite both sectors, the positive energy one and the negative energy one, without bound. In a quantum system this translates into an instability of the vacuum. The decay probability of the vacuum to a state involving negative energy excitations would be infinite, since the phase space is, and consequently no Lorentz invariant stable vacuum could exist. Looking at the Hamiltonian (23), the kinetic term is always positive and we can focus to constant field configurations. The potential is instead unbounded for $\lambda < 0$ when $(\Phi^\dagger\Phi) = const. \to \infty$, hence we need:

$$\lambda \geq 0.$$

Even if we obtained (22) by demanding $SU(2)$ invariance (and restricting to $d \leq 4$ terms), the resulting Lagrangian is invariant under a bigger symmetry group. First it is simple to check that $U(1)$ phase transformations $\Phi \to e^{i\alpha}\Phi$ are a symmetry, hence (22) is invariant *at least* under $U(1) \times SU(2) = U(2)$. In fact the symmetry group is even bigger. To see this let us write $\Phi^\dagger\Phi$ in terms of ϕ_i:

$$\Phi^\dagger\Phi = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^4 = \sum_{i=1}^{4} \phi_i^2.$$

Similarly we have

$$\partial_\mu\Phi^\dagger\partial^\mu\Phi = \sum_{i=1}^{4} \partial_\mu\phi_i\partial^\mu\phi_i.$$

We then can conclude that (22) is invariant under the group $O(4)$, under which $\phi_i \to \sum_{j=1}^{4} O_{ij}\phi_j$, $O \in O(4)$. Indeed

$$\sum_{i=1}^{4}\phi_i^2 \to \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}(O_{ij}\phi_j)(O_{ik}\phi_k) = \sum_{j=1}^{4}\sum_{k=1}^{4}\phi_j\underbrace{(O^TO)_{jk}}_{=\delta_{jk}}\phi_k = \sum_{j=1}^{4}\phi_j^2.$$

Invariance of the kinetic term follows in the same way.

Let us now add higher dimensional terms to (22) which respect the $U(2)$ symmetry. First it is easy to check that there are no dimension five terms which are both Lorentz and $U(2)$ invariant. At dimension 6 we have only one possibility with no derivatives

$$\mathcal{O}_1 = (\Phi^\dagger\Phi)^3.$$

At dimension six we can also write terms with two derivatives and four fields. Then we find

$$\begin{aligned}
&\mathcal{O}_2 = (\Phi^\dagger\Phi)(\partial_\mu\Phi^\dagger\partial^\mu\Phi),\\
&\mathcal{O}_3 = (\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi) + c.c. = (\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi) + (\partial_\mu\Phi^\dagger\Phi)(\partial^\mu\Phi^\dagger\Phi),\\
&\mathcal{O}_4 = i\left[(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi) - c.c.\right] = i(\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi) - i(\partial_\mu\Phi^\dagger\Phi)(\partial^\mu\Phi^\dagger\Phi),\\
&\mathcal{O}_5 = (\partial_\mu\Phi^\dagger\Phi)(\Phi^\dagger\partial^\mu\Phi).
\end{aligned}$$

Notice that $(\Phi^\dagger\partial_\mu\Phi)^* = (\partial_\mu\Phi^\dagger\Phi)$. Then these are found just taking all possible combinations of four fields where derivatives act on different fields and requiring reality. Terms where two derivatives act on the same field can be obtained from these adding a total derivative, hence we do not need to include them. For instance:

$$(\Phi^\dagger\Phi)(\Phi^\dagger\partial^2\Phi) + (\Phi^\dagger\Phi)(\partial^2\Phi^\dagger\Phi) = -2\mathcal{O}_2 - \mathcal{O}_3 - 2\mathcal{O}_5 + \partial_\mu\left[(\Phi^\dagger\Phi)(\Phi^\dagger\partial^\mu\Phi) + c.c.\right].$$

We neglect terms with four derivatives.
Now we can add to (22) these terms. Since $[\mathcal{L}] = 4$, the coupling in front the $d = 6$ terms must have dimension of an inverse mass square. Hence the modification of (22) induced by the addiction of these can always be written as

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} = \mathcal{L} + \frac{1}{M^2}\sum_{i=1}^{5}\lambda_i\mathcal{O}_i, \tag{24}$$

where the λ_i are dimensionless and M is a mass.
By construction (24) is still $U(2)$ invariant. Without doing further computations we can also easily see that $\mathcal{O}_1$ and $\mathcal{O}_2$ are $O(4)$ invariant, since they are built with the same building blocks present in (22). However it is possible to see that $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}_5$ are not $O(4)$ invariant. Consider indeed

$$\begin{aligned}
(\Phi^\dagger\partial_\mu\Phi) &= \phi_1\partial_\mu\phi_1 + i\phi_1\partial_\mu\phi_2 - i\phi_2\partial_\mu\phi_1 + \phi_2\partial_\mu\phi_2 + \phi_3\partial_\mu\phi_3 + i\phi_3\partial_\mu\phi_4 - i\phi_4\partial_\mu\phi_3 + \phi_4\partial_\mu\phi_4\\
&= \sum_{i=1}^{4}\phi_i\partial_\mu\phi_i + i\sum_{i,j=1}^{2}\epsilon_{ij}\phi_i\partial_\mu\phi_j + i\sum_{i,j=3}^{4}\epsilon_{ij}\phi_i\partial_\mu\phi_j.
\end{aligned}$$

Plugging this into the explicit expression of $\mathcal{O}_5$, for instance, we find

$$\mathcal{O}_5 = \left(\sum_{i=1}^{4}\phi_i\partial_\mu\phi_i\right)^2 + \left(\sum_{i,j=1}^{2}\epsilon_{ij}\phi_i\partial_\mu\phi_j + \sum_{i,j=3}^{4}\epsilon_{ij}\phi_i\partial_\mu\phi_j\right)^2.$$

The second term in the r.h.s. is not $O(4)$ invariant. We conclude that generically the modified Lagrangian (24) is $U(2)$ but not $O(4)$ invariant.