

Quantum Field Theory

Set 9

Exercise 1: Decomposition in Fourier modes

Given the decomposition of a real scalar field in a finite cubic volume $V = L^3$

$$\phi(t, \vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{\sqrt{V}} \phi_n(t) e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}} \quad (1)$$

Expand $\int d^3x (\vec{\nabla} \phi)^2(t, \vec{x})$ in Fourier modes $\phi_n(t)$.

Exercise 2: Hamiltonian in momentum space

Consider the Fourier expansions at a given time for a real scalar field $\phi(x, t)$ and its momentum $\pi(x, t)$:

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \phi(k, t), \quad \pi(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} \pi(k, t)$$

and the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2} \pi(x, t)^2 + \frac{1}{2} (\nabla \phi(x, t))^2 + \frac{1}{2} m^2 \phi(x, t)^2 \right]$$

- show that $\phi(k, t) = \phi^*(-k, t)$ and $\pi(k, t) = \pi^*(-k, t)$,
- compute H in terms of the Fourier transformed field and momentum.

Exercise 3: Ladder operators

Given the algebra of the ladder operators (non relativistic normalization)

$$[a(\vec{q}), a^\dagger(\vec{p})] = (2\pi)^3 \delta^3(\vec{q} - \vec{p}),$$

and the expression of the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k}),$$

compute the commutation relations

$$[H, a(\vec{p})] = ? \quad [H, a^\dagger(\vec{p})] = ?$$

Exercise 4: Invariant measure

Consider the measure on three dimensional momentum space $\frac{d^3k}{(2\pi)^3 2k_0}$, where $k_0 = \sqrt{|\vec{k}|^2 + m^2}$.

- Performing a boost in a particular direction, say \hat{k}_3 , show the invariance of the measure under Lorentz transformations.
(*Hint:* use the form of the boost transformations $k'_0 = \cosh(\eta) k_0 + \sinh(\eta) k_3$, $k'_3 = \cosh(\eta) k_3 + \sinh(\eta) k_0$.)

- Show that the distribution $(2\pi)^3 2p^0 \delta^3(\vec{k} - \vec{p})$ is invariant under Lorentz transformations that is

$$(2\pi)^3 2p^0 \delta^3(\vec{k} - \vec{p}) = (2\pi)^3 2(\Lambda p)^0 \delta^3(\vec{\Lambda k} - \vec{\Lambda p}) \quad (2)$$

Exercise 5: Noether's charge as generator of transformation: part 2

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the global symmetry transformation defined by

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu \\ \phi_a(x) &\longrightarrow \phi'_a(x) = D[\phi]_a(x) \simeq \phi_a(x) + i\alpha^i T_{i,a}^b \phi_b(x), \end{aligned}$$

where the $T_{i,a}^b$ are the representation of the generators of some Lie algebra of some Lie group \mathcal{G} .

- Find the following transformation of the conjugate momenta $\pi^a(x)$

$$\pi^a(x) \longrightarrow \pi'^a(x) = D^\pi[\pi]^a(x) \simeq \pi^a(x) - i\alpha^i T_{i,b}^a \pi^b(x),$$

and show that charge Q_i built starting from the Noether's current is the generator of the transformation:

$$\delta_\alpha \pi^a(x) \equiv \pi'^a(x) - \pi^a(x) = \alpha^i \{Q_i, \pi^a(x)\} = -i\alpha^i T_{i,b}^a \pi^b(x).$$

- Using the Jacobi identity and that $\{Q_i, \phi_a\} = iT_{i,a}^b \phi_b(x)$ (as you proved in the last exercise set), compute the following object

$$\{\phi_a \{Q_i, Q_j\}\}, \quad (3)$$

and deduce $\{Q_i, Q_j\}$.

- Compute explicitly the Poisson bracket

$$\{Q_i, Q_j\} \equiv \int d^3z \frac{\delta Q_i}{\delta \pi_a(z)} \frac{\delta Q_j}{\delta \phi^a(z)} - \frac{\delta Q_i}{\delta \phi_a(z)} \frac{\delta Q_j}{\delta \pi^a(z)} \quad (4)$$

and check that the result is consistent with what you have found before.

Exercise 6: Noether's current: a different approach

Notice that this exercise is covered in Sec. 3.3.1 of the lecture notes.

There is another way to find the Noether's current of a Lagrangian with a given symmetry without having to use the explicit formula of the theorem. This can sometimes be faster than applying the formula. The idea is that we consider a transformation of the symmetry group where we take the Lie parameters α dependent on x (local transformation). We know that if α does not depend on x (global transformation) then $\delta S = 0$ under this transformation by definition of symmetry. If α depends on x then δS will no longer be zero, but at the leading order in α it will be proportional to $\partial\alpha$. Explicitly

- Compute δS as in the derivation of the Noether theorem but being careful that α depends on x . Knowing that δS is zero when α is constant show that

$$\delta S = \int d^4x (\partial_\mu \alpha(x)) J^\mu + \mathcal{O}(\alpha^2)$$

where J^μ is the Noether current given by the Noether theorem (you can work in the case $K = 0$ for simplicity).

- Apply this procedure to the $U(1)$ symmetry in a complex scalar field theory and show that you get the same current as the previous exercise set.