Quantum Field Theory

Set 8

Exercise 1: Noether's currents in classical mechanics

Consider the general Lagrangian of a classical system

$$L = \sum_a \frac{m_a}{2}(\dot{\vec{q}}_a)^2 - \sum_{a\neq b} V\left(|\vec{q}_a - \vec{q}_b|\right) .$$

Using the Noether procedure compute the Noether current associated to the following transformations

- time translations:
$$t' = t - \alpha, \quad \vec{q}\,'_a(t') = \vec{q}_a(t)$$
- coordinate translations:
$$t' = t, \quad \vec{q}\,'_a(t) = \vec{q}_a(t) + \vec{\alpha}$$
- coordinate rotations:
$$t' = t, \quad q'\,^i_a(t) = \mathcal{R}^i_j q^j_a(t)$$

Exercise 2: Noether's current for $U(1)$ global symmetry

Consider the Lagrangian of a free *complex* massive scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi .$$

Show that the Lagrangian density is invariant under the following transformation

$$x_\mu \longrightarrow x_\mu , \qquad \phi(x) \longrightarrow e^{i\alpha}\phi(x),$$

where α does not depend on x.

- Compute the Noether current associated to this symmetry and show that is conserved only if one uses the equations of motion.
- How does the result change if we add to the Lagrangian an interaction term of the form $V(\phi\phi^\dagger)$?

Exercise 3: Noether's charge as generator of transformations

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the symmetry transformation defined by

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = f(x) \simeq x^\mu - \epsilon^\mu_i \alpha^i, \\ \phi_a(x) &\longrightarrow \phi'_a(x') = D[\phi]_a(f^{-1}(x')) \simeq \phi_a(x') + \alpha^i \Delta_{ai}(\phi(x')), \end{aligned}$$

- Show that charge Q_i built starting from the Noether's current is the generator of the transformation:
$$\delta_\alpha \phi_a(x) \equiv \phi'_a(x) - \phi_a(x) = \alpha^i \left\{Q_i, \phi_a(x)\right\} = \alpha^i \Delta_{ai}(x).$$
- Compute explicitly the charges associated to spacetime translation and Lorentz transformations in a scalar field theory.
- Using the formalism of the Poisson brackets check that the charges related to translations and spatial rotations generate the respective infinitesimal transformations.

Exercise 4: $SU(2)$ Noether's current

Consider a pair of complex scalar fields $\Phi = \binom{\phi_1}{\phi_2}$ transforming according to the representation $j = 1/2$ of a global $SU(2)$ symmetry (do not confuse this symmetry with the Lorentz transformations: they are completely uncorrelated):

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu, \\ \Phi(x) &\longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \\ \phi_a(x) &\longrightarrow \phi'_a(x') = \mathcal{U}_a{}^b\phi_b(x), \qquad a, b = 1, 2. \end{aligned}$$

where $\mathcal{U}$ is a $SU(2)$ matrix. Given the Lagrangian density

$$\mathcal{L} = \partial^\mu\Phi^\dagger\,\partial_\mu\Phi - m^2\Phi^\dagger\Phi - \frac{\lambda}{2}\left(\Phi^\dagger\Phi\right)^2,$$

- Show that the Lagrangian density is invariant under $SU(2)$ transformations.
- Compute the Noether current for this symmetry.
- Add to the above Lagrangian the interaction with a real *triplet* $\vec{A} = (A_1, A_2, A_3)$ of $SU(2)$

  $$\tilde{\mathcal{L}} = \mathcal{L} + \frac{1}{2}\partial^\mu A^T\,\partial_\mu A + A_i\Phi^\dagger\sigma^i\Phi.$$

  Is the new Lagrangian density invariant under $SU(2)$ transformations involving the field Φ and $\vec{A}$?
- If yes, what other terms with dimension less or equal than four can be added to the Lagrangian so that it remains invariant under $SU(2)$ (and the Lorentz group)?
  Hint: recall the following properties of Pauli matrices

  $$\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + 2i\varepsilon^{ijk}\sigma^k, \qquad \sigma^i_{\alpha\beta}\sigma^i_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}.$$

Exercise 5: WZW term

Consider 5 real scalar fields, η_i $(i = 1, 2)$ and ϕ_I $(I = 1, 2, 3)$, with Lagrangian

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}\partial_\mu\eta_i\partial^\mu\eta_i + \frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi_I - \frac{m^2}{2}(\eta_i\eta_i) - \frac{M^2}{2}(\phi_I\phi_I) \\ & - \lambda(\eta_i\partial_\mu\eta_j)(\partial_\nu\phi_I\partial_\rho\phi_J\partial_\sigma\phi_K)\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

where ϵ^{ij}, ϵ^{IJK}, $\epsilon^{\mu\nu\rho\sigma}$ are the Levi-Civita antisymmetric tensors in respectively 2, 3 and 4 dimensions.

- What is the dimensionality of the coupling λ?
- Find the symmetries of the system
  Hint: consider a generic transformation $\eta_i \to M_{ij}\eta_j$ and $\phi_i \to N_{ij}\phi_j$ where M and N are 2×2 and 3×3 matrices respectively. What properties should M and N satisfy to leave the Lagrangian invariant? Also remember that for a $N\times N$ matrix M we have the following identity

  $$\epsilon_{i_1,i_2,\ldots,i_N}M_{i_1j_1}M_{i_2j_2}\ldots M_{i_Nj_N} = \det(M)\epsilon_{j_1,j_2,\ldots,j_N}$$
- On which parameters do the Noether currents depend on?
- Suppose now that the η fields are complex, and the Lagrangian is:

  $$\begin{aligned} \mathcal{L} =& \frac{1}{2}\partial_\mu\eta_i^*\partial^\mu\eta_i + \frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi_I - \frac{m^2}{2}(\eta_i^*\eta_i) - \frac{M^2}{2}(\phi_I\phi_I) \\ & - i\lambda(\eta_i^*\partial_\mu\eta_i)(\partial_\nu\phi_I\partial_\rho\phi_J\partial_\sigma\phi_K)\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

  What are in this case the symmetries of the system?

- What if we change the interaction term to:

$$\lambda(\eta_i \partial_\mu \eta_j)(\partial_\nu \phi_I \partial_\rho \phi_J \partial_\sigma \phi_K)\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} + h.c.$$