Quantum Field Theory

Set 6

Exercise 1: The adjoint representation

Consider a Lie group $\mathcal{G}$ of dimension N. Let X^a be the generators of $\mathcal{G}$ in a generic representation $D(g)$ of dimension d that satisfy the Lie Algebra

$$[X^a, X^b] = if^{abc}X^c$$

Assume for simplicity to parametrise $D(g(\alpha)) = e^{iX^a\alpha^a}$ where $\alpha_{i=1,\dots,N}$ are the N parameters defining the group element. The N generators define the $N-$dimensional vector space of the Lie algebra, call it $\mathfrak{g}$. The adjoint representation $R(g)$ is defined on this vector space by the following action

$$R(g): \; V_{ij} = v^a X^a_{ij} \to_g V'_{ij} = D(g)_{ik}V_{kl}D(g^{-1})_{lj} \,. \tag{1}$$

- Show that if $V = v^a X^a$ then $V' = v'^a X^a$, i.e. the $R(g)$ defines a map from $\mathfrak{g}$ to itself (automorphism). *Hint: You can use the Hadamard formula*

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \dots$$

- Show that $R(g)$ is a representation. What is its dimension?
- The action of $R(g)$ is equivalent to the linear transformation on the components of V

$$v'^a = R^{ab}(g)v^b \,,$$

  Given $R(g(\alpha)) = e^{i\tilde{X}^a\alpha^a}$, expanding eq. (1) at the first order in α, show that the generators $\tilde{X}^a$ are related to the structure constants as

$$\tilde{X}^a_{ij} = -if_{aij}$$

- Show that the Jacobi identity for the Lie Algebra implies that the structure constant f satisfy the Algebra and so the $\tilde{X}^a$ are valid generators.

Exercise 2: building $SU(2)$ representations

Construct explicitly the following representation of $SU(2)$, i.e. write the matrix form of the generators for:

- $j = 1$ representation;
- $j = 3/2$ representation;
- $j = 2$ representation.

Recall that in set 5 you showed that a representation j is made of $2j+1$ vectors $|j,m\rangle$, with $m = -j, -j+1, \dots, j-1, j$, on which the Lie algebra is represented as:

$$T^3\,|j,m\rangle = m\,|j,m\rangle \,,$$
$$T^{\pm}\,|j,m\rangle = \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle \,,$$

$$T^{\pm} = \frac{T^1 \pm iT^2}{\sqrt{2}}.$$

Write explicitly a rotation around the $\hat{z}$ axis, i.e. $e^{i\phi T^3}$, in these representations.
Homework: construct explicitly the rotations $e^{i\phi T^1}$ and $e^{i\phi T^2}$. Use `Mathematica` if you want.

Exercise 3: $SO(3)$ vs $SU(2)$: the adjoint representation

In this exercise we will explore the connection between the two groups $SO(3)$ and $SU(2)$.

Consider the representation of the $SO(3)$ group on the three dimensional vector space $\mathbb{R}^3$ (also known as the *defining representation*).

- Show that the set of three 3×3 matrices $(T^a)_i{}^j = -i\epsilon_{aij}$ $(a = 1, 2, 3)$ is a basis for (a representation of) the Lie Algebra of $SO(3)$, and find this algebra explicitly (i.e. find the structure constants of $so(3)$).
- Consider an element of the group $R(\vec{\alpha}) = e^{i\alpha^a T^a}$. Write $\vec{\alpha} = \theta\,\vec{n}$ with $\vec{n} = \vec{\alpha}/|\vec{\alpha}|$. Expanding for $\theta \ll 1$, find how a vector $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ transforms under the action of an infinitesimal element of the group ($R(\vec{\alpha})$ represents a rotation around the $\vec{n}$ direction by an angle θ).
- Consider for simplicity the case $\vec{n} = \vec{e}_3$. Use the exponential map to find the group transformation for a finite value of θ.

Take the group $SU(2)$. Consider the tensor product representation of this group on the vector space of traceless Hermitean matrices:

$$V = \left\{ M \in M(2 \times 2, \mathbb{C}) | M = M^\dagger,\ \mathrm{Tr}[M] = 0 \right\}.$$

The action of an element U of the group (recall that U is itself a 2×2 matrix) is given by

$$U : M \to UMU^\dagger.$$

- Show that the three Pauli matrices σ^i form a basis of the space V and therefore any $M \in V$ can be written as $M = \sum_{i=1}^{3} y_i \sigma^i$.
- Show that the vector space V coincides with the vector space of the $su(2)$ Lie Algebra.
- Consider now an infinitesimal $SU(2)$ transformation on this space. Show that the action of a given element U of $SU(2)$ corresponds to a rotation of the three dimensional vector $\vec{y} \equiv (y^1, y^2, y^3)$. Is this a faithful representation for $SU(2)$?

Exercise 4: Tensor product of $SU(2)$ vectors

- Using the highest-weight technique reduce the tensor product of two spin-1 $SU(2)$ representations.
- An $SU(2)$ vector can be thought as a 1-index tensor v_i transforming as $v_i \to \sum_j R_{ij} v_j$, where R is an orthogonal matrix. Starting from the product of 2 vectors $v_i w_j$, show that this is a representation of $SU(2)$: the tensor product of 2 vector representations. Show that it is reducible and do the reduction in terms of irreducible tensor representations of $SU(2)$. Show that the result matches with what you found by the highest weight technique.

Exercise 5: Lorentz Group and Lorentz Algebra

Consider the set of the Lorentz transformations in space-time. Show that they correspond to the group

$$O(1,3) = \left\{ \Lambda \in GL(4, \mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \right\},$$

where η is the metric with Minkowski signature $\eta = \mathrm{diag}(1, -1, -1, -1)$. Starting from the above definition

- identify the Lie algebra;
- compute the dimension of the Lie algebra;

A basis of this Lie algebra is provided by the following matrices:

$$(\mathcal{J}^{\mu\nu})^\rho{}_\sigma = i\,(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)\,;$$

- Show that Lie algebra structure is:

$$\left[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}\right] = i(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha});$$

- define the quantities

$$J^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk}\,, \qquad K^i = \mathcal{J}^{i0}\,,$$

  and compute the commutation relations between them:

$$\left[J^i, J^j\right] =?\,, \qquad \left[J^i, K^j\right] =?\,, \qquad \left[K^i, K^j\right] =?\,.$$

Can you guess which types of transformations do the vectors J^i and K^i generate?

Exercise 6: Lorentz boosts and rapidity

Consider a Lorentz boost along the x-axis

$$\Lambda = \exp[-i\eta\mathcal{J}^{10}].$$

The parameter η is called *rapidity* of the boost along that axis. Show that this boost can be written as

$$\Lambda = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) & 0 & 0 \\ -\sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Express the velocity β and the boost factor γ in terms of the rapidity.
Show that applying two boosts along the same direction, characterized by rapidities η and η', the total transformation is again a boost along the same direction, characterized by rapidity $\eta + \eta'$.