

Quantum Field Theory

Set 4

Exercise 1: Schur's lemma

Suppose that there are two irreducible inequivalent representations D_1 and D_2 . Let A be a linear operator which commutes with the direct sum of these two representations $D_1 \oplus D_2$. Show that $A = \lambda_1 \mathbb{1} \oplus \lambda_2 \mathbb{1}$.

Exercise 2: Jacobi Identity

Show that, for any Lie Algebra defined by the commutation relations

$$[T^a, T^b] = i f^{abc} T^c,$$

the following *Jacobi identity* holds:

$$\sum_d (f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd}) = 0.$$

This is a straightforward consequence of the identity:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0.$$

Exercise 3: The Lie theorem

Consider a general Lie group \mathcal{G} parametrized in a neighborhood of the identity by some choice of coordinates $\alpha \in \mathbb{R}^n$ so that the origin parametrizes the identity element $g(0) = e$. Given two group elements $g(\alpha)$ and $g(\beta)$, we can define a function $p(\alpha, \beta)$ that implements the product rule of the group

$$g(\alpha)g(\beta) = g(p(\alpha, \beta)).$$

The manifold structure of the Lie group tells us that this function is smooth, and in particular it can be expanded in series in a finite neighborhood of the identity.

- Using the group properties compute $p(\alpha, 0)$ and $p(0, \beta)$.
- Using the previous result, write the most general expansion of $p(\alpha, \beta)$ up to the second order in α and β (included).
- Show that the symmetric part of the coefficient of the second order in the expansion can be eliminated by a suitable change of parametrization $\alpha^i \rightarrow \alpha^i + \delta_{ab}^i \alpha^a \alpha^b + O(\alpha^3)$, while the antisymmetric part can't.
- Call $\bar{\alpha}$ the coordinate of the inverse element of α (in formulas $g(\alpha)^{-1} = g(\bar{\alpha})$). Find an expression for $\bar{\alpha}(\alpha)$ up to the second order in α (again included).
- Consider the following product $g(\alpha)^{-1}g(\beta)^{-1}g(\alpha)g(\beta) \equiv g(c(\alpha, \beta))$. Compute $c(\alpha, \beta)$ up to the second order in α and β and show that it satisfies the antisymmetry property of the Lie product.
- **Bonus (hard)**: use the associativity of the product to show that the Lie product satisfies the Jacobi identity. *Hint*: you will have to go to the third order in the expansion.

Exercise 4: Permutation group \mathcal{S}_3

Consider the set of all permutations of the configuration (a, b, c) . This is known as the permutation group \mathcal{S}_3 .

- What is the order $|\mathcal{S}_3|$ of the group?
- Represent the objects $\{1, 2, 3\}$ as vectors in \mathbb{R}^3 ,

$$|1\rangle = (1, 0, 0), \quad |2\rangle = (0, 1, 0), \quad |3\rangle = (0, 0, 1).$$

Write the corresponding matrix representation of \mathcal{S}_3 . You should write $|\mathcal{S}_3|$ matrices.

- Show that the representation you constructed is *reducible*. In particular, can you guess non-trivial $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $\alpha_1 |1\rangle + \alpha_2 |2\rangle + \alpha_3 |3\rangle$ remains invariant under any permutation? The vector space spanned by this vector is an *invariant subspace*.
- Find the orthogonal vector space. Show that is also an invariant subspace.
- Compute the *equivalent* representation to the one you constructed in the basis that spans each invariant subspace. Verify that it is block diagonal, where each block is irreducible. This shows that the original representation is *completely reducible*.