

Quantum Field Theory I

Set 14

Exercise 1: The Pauli–Lubanski (pseudo)vector (repeated from previous problem set; here for completeness).

The Poincare group has two casimirs, P^2 and W^2 , where P^μ and W^μ are respectively the momentum and the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma.$$

- Show that this definition is equivalent to

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}.$$

- Calculate and express the result of the following commutators in terms of the Pauli-Lubanski vector
 1. $P_\mu W^\mu$,
 2. $[P^\mu, W^\nu]$,
 3. $[J^{\mu\nu}, W^\rho]$ (Hint: Use the identity $\eta^{\mu\nu} \epsilon^{\rho\alpha\beta\sigma} = \eta^{\mu\rho} \epsilon^{\nu\alpha\beta\sigma} + \eta^{\mu\alpha} \epsilon^{\rho\nu\beta\sigma} + \eta^{\mu\beta} \epsilon^{\rho\alpha\nu\sigma} + \eta^{\mu\sigma} \epsilon^{\rho\alpha\beta\nu}$),
 4. $[W^\mu, W^\nu]$.
- Show that W^2 is a Casimir of the Poincare group.

Exercise 2 (Optional): The Casimirs of the Poincaré group

- Consider the generators of the Poincaré group, i.e. the translation generators P_μ and the Lorentz generators $J_{\mu\nu}$ (which are antisymmetric in $\mu \leftrightarrow \nu$). Let us write $J^{\mu\nu}$ (in analogy to electromagnetism) as

$$J^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad (1)$$

By applying a series of Lorentz transformations to P and J , show that they can be brought to the form

$$J^{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix} \quad P^\mu = \begin{pmatrix} P^0 \\ 0 \\ P^2 \\ 0 \end{pmatrix} \quad (2)$$

Hint: Argue using the electromagnetism analogy that one can apply some rotations and boosts in such a way to make $\vec{E} \rightarrow (E, 0, 0)$ and $\vec{B} \rightarrow (B, 0, 0)$. Argue that after having fixed B and E , there are two boosts left that one can use to set $P^1 \rightarrow 0$ and $P^3 \rightarrow 0$.

- Since we exhausted all Lorentz transformations to make most components vanish, this construction shows that there are 4 quantities that are invariant under Lorentz transformations. It is useful to write them in a more covariant manner. To achieve this compute

$$C_1 = J^{\mu\nu} J_{\mu\nu} \quad C_2 = \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \quad C_3 = P^\mu P_\mu \quad C_4 = W^\mu W_\mu \quad (3)$$

using (2). Conclude that C_1, C_2, C_3 and C_4 contain as much information as E, B, P^0, P^2 but have the advantage of being manifestly Lorentz invariant.

- Finally, show that C_1 and C_2 do not commute with translations, while C_3 and C_4 do. Thus, C_3 and C_4 are the two Casimirs of the Poincaré group.

Exercise 3: The spin of Dirac Fermion states

In this exercise you will show that Dirac Fermion states have spin 1/2.

Consider the state of a single Dirac fermion

$$|\Psi\rangle = \int d^3x f_\alpha(x) \psi_\alpha^\dagger(x) |0\rangle. \quad (4)$$

Here ψ is the Dirac field and f is a generic wave packet. Using the mode expansion of the field

$$\psi_\alpha(x) = \sum_r \int d\Omega_p u(p, r) e^{-ipx} a_r(p) + v(p, r) e^{ipx} b_r^\dagger(p), \quad (5)$$

rewrite the state as

$$|\Psi\rangle = \sum_r \int d\Omega_p F(p, r) |p, r\rangle, \quad (6)$$

where

$$F(p, r) = u_\alpha^\dagger(p, r) \tilde{f}_\alpha(p), \quad (7)$$

and

$$\tilde{f}_\alpha(p) = \int d^3x e^{ipx} f_\alpha(x). \quad (8)$$

Consider the angular momentum operator you computed in Exercise 14.2

$$J^i = \int d^3x \psi_\alpha^\dagger(t, \vec{x}) \left([\vec{x} \wedge (-i\vec{\nabla})] \delta_{\alpha\beta} + (\Sigma)_{\alpha\beta} / 2 \right) \psi_\beta(t, \vec{x}). \quad (9)$$

Looking at this operator, one might be tempted to identify the first term as the orbital angular momentum and the second one as the spin. However, this is not true and the action of the two operators on the state $|\Psi\rangle$ will mix. Show that

$$\vec{J} |\Psi\rangle = \sum_{r, r'} \int d\Omega_p \left((-i\vec{p} \wedge \vec{\partial}_p) \delta_{rr'} + (\vec{\sigma})_{rr'} / 2 \right) F(p, r') |p, r\rangle. \quad (10)$$

Only at this point you can identify the first term as the representation of the orbital angular momentum and the second one as the spin acting on the wave function $F(p, r)$. Choosing a state with zero orbital angular momentum, realize that Dirac fermion states have spin 1/2.

Hint: You may find convenient to first compute the commutator of the angular momentum operator \vec{J} with the field ψ^\dagger

$$[\vec{J}, \psi_\alpha^\dagger(x)] = \psi_\beta^\dagger(t, \vec{x}) \left(\vec{x} \wedge (i\vec{\nabla}) + \vec{\Sigma} / 2 \right)_{\beta\alpha}. \quad (11)$$

Then, you should prove

$$u^\dagger(p, r) \vec{\Sigma} / 2 = (-i\vec{p} \wedge \vec{\partial}_p) u^\dagger(p, r) + \sum_{r'} (\vec{\sigma})_{rr'} / 2 u^\dagger(p, r'). \quad (12)$$

Exercise 4: Transformation of the boost generators under translations

Starting from the group composition rule:

$$g(\Lambda_1, a_1) g(\Lambda_2, a_2) = g(\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1), \quad (13)$$

with $g(\Lambda, a) \in ISO(3, 1)$, prove the following transformation of $J^{\mu\nu}$:

$$e^{-iaP} J^{\mu\nu} e^{iaP} = J^{\mu\nu} + P^\nu a^\mu - P^\mu a^\nu. \quad (14)$$

where a^μ is a translation vector.