Quantum Field Theory

Set 13

Exercise 1: U(1) symmetry and chiral symmetry

Given the Lagrangian density of a massless Dirac fermion $\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$:

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\psi,$$

verify that it is invariant under a symmetry $U(1)_L \times U(1)_R$ where each $U(1)$ acts independently on the left or right component of the Dirac fermion.
Compute the Noether's currents J^μ_L and J^μ_R associated to these symmetries.
Consider the combinations

$$J^\mu_V = J^\mu_R + J^\mu_L, \qquad\qquad J^\mu_A = J^\mu_R - J^\mu_L.$$

Show that these are the Noether's currents associated to the following symmetry transformation acting on the Dirac fermion:

$$\begin{aligned} U(1)_V: &\quad \psi \longrightarrow e^{i\alpha}\psi, \\ U(1)_A: &\quad \psi \longrightarrow e^{i\beta\gamma^5}\psi, \end{aligned}$$

where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.
What changes if one adds a mass term for the Dirac fermion: $m\bar{\psi}\psi$?

Exercise 2: Angular momentum in the Dirac theory

Starting from the Dirac Lagrangian (written in its hermitian form) compute the Noether's current $M^\rho_{\mu\nu}$ associated to Lorentz invariance.
Introduce the angular momentum operator

$$J^k \equiv \frac{1}{2}\epsilon^{ijk}\int d^3x\; M^0_{ij}.$$

Show that it can be written as

$$J^k = \int d^3x\; \psi^\dagger(t,\vec{x})(L_k + \Sigma_k/2)\psi(t,\vec{x}),$$

where

$$\begin{aligned} L^k &= [\vec{x} \wedge (-i\vec{\nabla})]^k, \\ \Sigma^k &= \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \end{aligned}$$

are respectively the orbital and spin parts.
Defining

$$\vec{S} = \int d^3x\; \psi^\dagger(t,\vec{x})\frac{\vec{\Sigma}}{2}\psi(t,\vec{x}),$$

compute the eigenvalue of the operator $\vec{S}^2$ on a generic one-particle state in position space, $\psi^\dagger_\alpha(t,\vec{x})|0\rangle \equiv |x,\alpha\rangle$, and show this way that $\psi^\dagger$ creates states with spin one half.

Exercise 3: Commutator of bilinear operators

Start from the following general identities (which you can easily check explicitly):

$$[A, BC] = \{A, B\}\, C - B\, \{A, C\}$$

$$[A, BC] = B\, [A, C] + [A, B]\, C$$

Consider a bilinear operator $O_A = \psi_i^\dagger A_{ij} \psi_j$ where ψ_i can be either bosonic variables:

$$[\psi_i, \psi_j^\dagger] = \delta_{ij}\,, \quad [\psi_i, \psi_j] = 0$$

or fermionic variables:

$$\left\{\psi_i, \psi_j^\dagger\right\} = \delta_{ij}\,, \quad \{\psi_i, \psi_j\} = 0.$$

Compute the commutators $[\psi_i^\dagger, O_A]$ and $[\psi_i, O_A]$. Use this result to argue that the equations of motion for a Dirac field in Hamiltonian formalism do not depend on whether we quantize the field using commutation or anticommutation relations. Namely, given the Hamiltonian operator

$$H = \int d^3x \bar{\psi}\, (-i\vec{\gamma} \cdot \nabla + m)\, \psi,$$

show that

$$i\frac{\partial \psi(\vec{x}, t)}{\partial t} = [\psi(\vec{x}, t), H], \qquad i\frac{\partial \pi(\vec{x}, t)}{\partial t} = [\pi(\vec{x}, t), H]$$

give the same equations of motion both if you impose $[\psi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})$ or $\{\psi(\vec{x}, t), \pi(\vec{y}, t)\} = i\delta^3(\vec{x} - \vec{y})$, where $\pi(\vec{x}, t) = i\psi^\dagger(\vec{x}, t)$.

Consider now two bilinear operators $O_A = \psi_i^\dagger A_{ij} \psi_j$, $O_B = \psi_i^\dagger B_{ij} \psi_j$. Compute the commutators relations $[O_A, O_B]$ both in case the variables ψ are bosonic and in case they are fermionic.

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References & Exercises

- "A Modern Introduction to Quantum Field Theory", Maggiore:
  Paragraphs: 1.2, 2.1-2.7, 3.1-3.4, 4.1-4.2.2
- "An Introduction to Quantum Field Theory", Peskin & Schroeder:
  Pages: 13-26, 35-62

- "An Introduction to Quantum Field Theory", Peskin & Schroeder:
  Problems **2.1**, **3.4**, **3.5**
- "A Modern Introduction to Quantum Field Theory", Maggiore:
  Problems **2.1-2.5**, **3.1**, **3.5**, **3.6** (solutions at the end of the book)