Quantum Field Theory

Set 12

Exercise 1: Solution of Dirac equation

Given the solution of the Dirac equation $(\not{\tilde{p}} - m)u_0 = 0$ in the rest frame of the fermion, $\tilde{p}^\mu = (m, 0, 0, 0)$,

$$u_0 = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

- Show that the boosted spinor $u(p) = \Lambda_D u_0$ is a solution of the Dirac equation in a generic frame $p^\mu = (E, p^1, p^2, p^3)$, $p_\mu p^\mu = m^2$:
$$(\not{p} - m)u(p) = 0,$$
where Λ_D is the representation of the Lorentz transformation Λ^μ_ν that connects the rest frame with the generic frame: $p^\mu = \Lambda^\mu_\nu \tilde{p}^\nu$.
- Start from the simple case $p^\mu = (E, 0, 0, p^3)$, write the explicit form of Λ^μ_ν in terms of $\cosh(\eta)$ and $\sinh(\eta)$ and derive:
$$e^{\pm\eta} = \frac{E \pm p^3}{m}. \tag{1}$$
- Starting from the general representation of a Lorentz transformation in the Dirac representation
$$\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}(i\theta^i + \eta^i)\sigma^i} & 0 \\ 0 & e^{-\frac{1}{2}(i\theta^i - \eta^i)\sigma^i} \end{pmatrix},$$
show that
$$\Lambda_D = \begin{pmatrix} \cosh\left(\frac{\eta}{2}\right) 1_2 - \sinh\left(\frac{\eta}{2}\right)\sigma^3 & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) 1_2 + \sinh\left(\frac{\eta}{2}\right)\sigma^3 \end{pmatrix}.$$
- Substitute the expression (1) for η and show that
$$u(p) = \begin{pmatrix} \sqrt{E - p^3\sigma^3} & 0 \\ 0 & \sqrt{E + p^3\sigma^3} \end{pmatrix} \frac{u_0}{\sqrt{m}}$$
- Starting form $p^\mu = (E, 0, 0, p^3)$ and then generalizing to arbitrary spatial momenta, show that $u(p)$ can be written in the more convenient form:
$$u(p) = \begin{pmatrix} \left(\left(\sqrt{E + m}\right) \mathbb{1}_2 - \frac{p^i\sigma^i}{\sqrt{E+m}} \right) \frac{\xi}{\sqrt{2}} \\ \left(\left(\sqrt{E + m}\right) \mathbb{1}_2 + \frac{p^i\sigma^i}{\sqrt{E+m}} \right) \frac{\xi}{\sqrt{2}} \end{pmatrix}.$$

Exercise 2: Basis of 4×4 matrices

Show that the following set of matrices:

$$\left\{ \mathbb{1},\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ \gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \right\}$$

is a basis for the space of 4×4 complex matrices, and in addition that they have definite transformation properties under the Lorentz group in the following sense: for a given Lorentz transformation ${\Lambda^\mu}_\nu$ one can take the Dirac representation Λ_D and consider

$$\Lambda_D^{-1} \Gamma \Lambda_D = \Gamma',$$

where Γ is one of the element of the above basis; then Γ and Γ' are related by a Lorentz transformation in the proper representation (for example: if $\Gamma \equiv \gamma^\mu$ then $\Gamma' \equiv {\Lambda^\mu}_\nu \gamma^\nu$, and so on).

Exercise 3: matrix elements

Consider the positive and negative energy solutions $u(p)$ and $v(p)$ of the Dirac equation, given by

$$u^s(\vec{p}) = \left(\begin{array}{c} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{array} \right) , \qquad v^s(\vec{p}) = \left(\begin{array}{c} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{array} \right) ,$$

where $p^\mu = (\sqrt{\vec{p}^2+m^2}, \vec{p})$, $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$ and $s = 1, 2$ corresponds to the different polarizations:

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

- Show that
$$\bar{u}^r(\vec{p})\gamma^\mu u^s(\vec{p}) = \bar{v}^r(\vec{p})\gamma^\mu v^s(\vec{p}) = 2p^\mu\delta^{rs} .$$
Conclude that the scalar products $u^{r\,\dagger}(\vec{p})u^s(\vec{p})$ and $v^{r\,\dagger}(\vec{p})v^s(\vec{p})$ are not Lorentz invariant.
- Prove that
$$\bar{u}^r(\vec{p})u^s(\vec{p}) = -\bar{v}^r(\vec{p})v^s(\vec{p}) = 2m\delta^{rs} , \qquad \bar{u}^r(\vec{p})v^s(\vec{p}) = \bar{v}^r(\vec{p})u^s(\vec{p}) = 0 .$$
Show that the last equation is equivalent to $u^{r\,\dagger}(\vec{p})v^s(-\vec{p}) = v^{r\,\dagger}(\vec{p})u^s(-\vec{p}) = 0$.
- Finally, prove the relations
$$\sum_{r=1,2} u^r(\vec{p})\bar{u}^r(\vec{p}) = \not{p} + m , \qquad \sum_{r=1,2} v^r(\vec{p})\bar{v}^r(\vec{p}) = \not{p} - m .$$