Quantum Field Theory

Set 10

Exercise 1: Heisenberg representation

Write the expansion of a free massive real scalar field in the Heisenberg representation:

$$\phi(x) \equiv \phi(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0}[a(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}] \tag{1}$$

where $a(\vec{k},t) = e^{-ik_0 t}a(\vec{k})$ and $k_0^2 = |\vec{k}|^2 + m^2$.

- Check that $\phi(x)$ defined by (1) satisfies the Klein-Gordon equation
- Starting from the commutation relations for the ladder operators:

  $$[a(\vec{k}), a^\dagger(\vec{p})] = (2\pi)^3 2k_0 \delta^3(\vec{k}-\vec{p})$$

  prove the canonical equal-time commutation relations:

  $$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0, \qquad [\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}),$$

- Argue that $[\phi(x), \phi(y)] = 0$ for x, y space-like separated: $(x-y)^2 < 0$. This is called the *microcausality* condition.

Exercise 2: Time independence of Noether charges

Consider the Lagrangian for a massive real scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

The Noether current related to space-time translations is the energy-momentum tensor, which in this theory takes the form:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}$$

Consider the charges related to the symmetries:

- Space-time translations: $P_\mu = \int d^3x : T_{0\mu} :$
- Boosts: $K_i = \int d^3x\,(x_0 : T_{0i} : - x_i : T_{00} :)$
- Rotations $J_{ij} = \int d^3x\,(x_i : T_{0j} : - x_j : T_{0i} :)$

Using the representation (1) for the operator ϕ and expressing the charges in terms of ladder operators, check explicitly that they don't depend on time, as expected.

Notation: The colons wrapped around an operator like this: $: O :$ denote the normal-ordering prescription. This means that when expressing O in terms of ladder operators, all the creation operators are put on the left of the annihilation operators, e.g.: $: a(\vec{q})a(\vec{k})^\dagger := a(\vec{k})^\dagger a(\vec{q})$.

Exercise 3: Lorentz invariance of Noether charges

Consider a generic field theory in $1+1$ dimensions. A Noether current $J^\mu(t,x)$ is associated to each symmetry. Then, given a coordinate system $\{t,x\}$, we define the Noether charge integrating the zero component of the current in space at fixed time $t=0$:

$$Q = \int dx\, J^0(0,x).$$

Since the Noether current satisfies $\partial_\mu J^\mu = \partial_t J^0 + \partial_x J^1 = 0$, Q is independent on time. This definition however is not manifestly Lorentz invariant. Indeed consider a different coordinate system $\{t',x'\}$ related to the previous one through a boost transformation:

$$t' = \gamma(t-\beta x), \qquad\qquad x' = \gamma(x-\beta t).$$

In these coordinates, we would naturally define the Noether charge integrating in space at fixed $t' = \gamma(t-\beta x) = 0$:

$$Q' = \int dx'\, {J'}^0(0,x'),$$

where ${J'}^\mu(t',x') = \Lambda^\mu_\nu J^\nu(t,x)$ is the Noether current in the second reference frame.

- Show that $Q' = Q$ (*Hint*: integrate the identity in $\partial_\mu J^\mu = 0$ in the plane section enclosed by the lines $t=0$ and $t'=0$).

Exercise 4: Noether charges as generators

Given the canonical commutation rules at equal time:

$$[\phi(\vec{x},t),\, \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}),$$
$$[\phi(\vec{x},t),\, \phi(\vec{y},t)] = [\pi(\vec{x},t),\, \pi(\vec{y},t)] = 0$$

- Show explicitly that the charge J_{ij} is the generator of spatial rotations for the field ϕ, i.e. that $[J_{ij},\phi(\vec{x},t)] = i\Delta_{ij}(\vec{x},t)$, where Δ_{ij} is the infinitesimal variation of the field under rotations:

$$\phi'(x) = \phi(x) + \Delta_{ij}(x)\alpha^{ij}$$

- Repeat for the boosts (generated by K_i) and space-time translations (generated by P_i).
- From the previous results deduce $[[J_{ij},P_k],\phi(\vec{x},t)]$. Is the result consistent with the Jacobi identity?

Exercise 5: Charged scalar field

The Lagrangian for a free complex scalar field reads:

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

or, equivalently:

$$\mathcal{L} = \mathcal{L}_{KG}[\phi_1] + \mathcal{L}_{KG}[\phi_2]$$

where $\mathcal{L}_{KG}$ is the Klein Gordon Lagrangian for a free real scalar field and ϕ_1 and ϕ_2 are defined through:

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

Upon quantization, the operator ϕ can be expressed as:

$$\phi(\vec{x},t) = \int d\Omega_k \left(e^{i\vec{k}\cdot\vec{x} - ik^0 t} a(\vec{k}) + e^{-i\vec{k}\cdot\vec{x} + ik^0 t} b^\dagger(\vec{k}) \right)$$

where:

$$a(\vec{k}) = \frac{a_1(\vec{k}) + i a_2(\vec{k})}{\sqrt{2}}, \qquad b(\vec{k}) = \frac{a_1(\vec{k}) - i a_2(\vec{k})}{\sqrt{2}}$$

- Find the commutation relation for the ladder operators $a(\vec{k})$, $b(\vec{k})$ (and their conjugates) given those of the operators $a_1(\vec{k})$, $a_2(\vec{k})$
- Express the (normal-ordered) Hamiltonian in terms of the ladder operators $a(\vec{k})$, $b(\vec{k})$
- Write the (normal-ordered) charge $Q = \int d^3x : J^0 :$ related to the $U(1)$ symmetry in terms of the operators $a(\vec{k})$, $b(\vec{k})$. (Recall that $J^\mu = i\left((\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right)$).
- What is the total charge of the state $|\psi\rangle = a^\dagger(\vec{k_1}) \dots a^\dagger(\vec{k_n}) b^\dagger(\vec{q_1}) \dots b^\dagger(\vec{q_m})|0\rangle$?

Exercise 6: Spontaneous symmetry breaking

Consider the following Lagrangian density for a charged scalar field ϕ:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi) \tag{2}$$

where

$$V(\phi) = m^2 \phi^* \phi + \frac{\lambda}{4}(\phi^* \phi)^2 .$$

- For $m^2 > 0$, $\lambda > 0$ show that the ground state is $\phi = 0$. Convince yourself that the system has two massive degrees of freedom.
- For $m^2 < 0$, $\lambda > 0$ show that the ground state is at $|\phi| = v$, where v is a constant. Find v in terms of m^2 and λ.
- Write $\phi = v + (\phi_1 + i\phi_2)/\sqrt{2}$ and substitute in the Lagrangian density. Show that there are one massive and one massless degrees of freedom. Find the mass of the massive mode.
- Repeat the last point for the parametrization $\phi = \frac{v+h}{\sqrt{2}} e^{i\pi}$, where now the fields are h and π. How does the Lagrangian looks like? Which form does the $U(1)$ symmetry of (2) take in this parametrization?