Quantum Field Theory

Homework 1

Exercise 1: Poincaré algebra on fields

Consider a scalar field $\phi(x)$. As shown in class, the Poincaré generators are represented on scalar fields by

$$P^\mu = i\partial^\mu, \qquad J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{1}$$

Compute the commutators

$$[P^\mu, P^\nu] = ?, \qquad [P^\mu, J^{\rho\sigma}] = ?, \qquad [J^{\mu\nu}, J^{\rho\sigma}] = ?, \tag{2}$$

using this representation. Show that you get the Poincaré algebra.

Exercise 2: Representation theory in Lorentz and internal symmetries

- In the first part of the exercise, you will study the **Lie algebra of the Lorentz group** and its **irreducible representations**, and apply these concepts to various tensor structures.
  1. Decompose an arbitrary rank-2 tensor $A_{\mu\nu}$ into irreducible representations of the Lorentz group.
  2. Determine the decomposition of the antisymmetric product of two vectors, $A_\mu B_\nu - A_\nu B_\mu$, into irreducible Lorentz representations.
- Consider two scalars ψ_I and ϕ_I, $I = 1, 2, 3$ transforming as triplets under an internal $SU(2)$, called Isospin.
  1. Decompose the product of $\phi_I \psi_J$ into irreducible representations of $SU(2)$,
  2. Given a third scalar χ_{IJ}, what constraint should it satisfy for it to correspond to Isospin 2?

Exercise 3. For Fun: Higher derivative terms in the Lagrangian

Part 1: generalized equations of motions

Consider a general Lagrangian[1] for a particle on a line $q(t)$:

$$L(q, \dot{q}, \ddot{q}, \ldots).$$

- Find the equation of motion (EOM). What is the order of the EOM in general? How many initial conditions are needed to solve it?

Part 2: $\ddot{q}^2$ theory

Consider

$$L(q, \dot{q}, \ddot{q}) = \frac{1}{2}\dot{q}^2 - V(q) + \frac{1}{2\Lambda^2}\ddot{q}^2. \tag{3}$$

- Find the equation of motion.
- Show that (3) is equivalent to the following two coordinates Lagrangian (i.e. it has the same equations of motions)

$$L'(q, \dot{q}, q_1 \dot{q}_1) = \frac{1}{2}\dot{q}^2 - V(q) + \dot{q}\dot{q}_1 - \frac{\Lambda^2}{2} q_1^2. \tag{4}$$

  Redefine coordinates q and q_1 in $L'(q, \dot{q}, q_1 \dot{q}_1)$ to get a diagonal kinetic term.
- *Optional:* compute the Hamiltonian associated to L' and show that it is not positive definite.

[1] Assume that L is written in order to be of lowest order in derivatives; for instance $q\ddot{q} = -\dot{q}^2 +$ total derivative.

Part 3: general higher derivative terms

Suppose you are given a Lagrangian $L(q, \dot{q}, \ddot{q})$. Then consider the following system which depends on q and two additional coordinates λ and q_1

$$L'(q, \dot{q}, q_1, \lambda) = L(q, \dot{q}, q_1) - \lambda(q_1 - \ddot{q})$$

- Use the EOM for λ to show that L' is equivaent to L.
- Use the EOM for q_1 to show that the same system can be described in terms of a Lagrangian $L''(q, \dot{q}, \lambda, \dot{\lambda})$ of two coordinates and their first derivatives. Do this explicitly in the Lagrangian (3) given in part 2 and recover the form (4).
- Find a generalization of this argument to show that any Lagrangian which depends on q and its derivatives up to order n, i.e. $L = L(q, \dot{q}, \ldots, q^{(n)})$, can always be rewritten in terms of a system which depends on q and other $n-1$ coordinates $\{\lambda_i\}$ and their first derivatives :

$$L(q, \dot{q}, \ldots, q^{(n)}) \longleftrightarrow L''(q, \dot{q}, \lambda_1, \dot{\lambda}_1, \ldots, \lambda_{n-1}, \dot{\lambda}_{n-1}).$$

  Hint: add $n-1$ Lagrange multipliers λ_i and $n-1$ coordinates q_i.