Quantum Field Theory

Homework Set 3

Exercise 1: Isospin × Lorentz spinors

Consider a Weyl spinor transforming in the representation $(1/2, 0)$ of the Lorentz group ψ_L:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$
$$\psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1}x'),$$

where Λ_L is the Lorentz transformation in the representation $(1/2, 0)$. We have omitted the spinor index α for shortness.
Identify the representation of the Lorentz group the following term belongs to:

$$\psi_L^\dagger \bar{\sigma}^\mu \psi_L.$$

Consider now a pair of *left* Weyl spinors: ψ_L^1, ψ_L^2 and assume they form a dublet Ψ_L of an additional *Isospin* $SU(2)$ symmetry (don't confuse this $SU(2)$ with the Lorentz Group, it's an internal symmetry). In addition take a single *right* Weyl spinor ψ_R which is a singlet of the $SU(2)$ Isospin (ψ_R, Ψ_L are completely unrelated fields, they are not obtained one from the other using ε). Note that the doublet $\Psi_L = \binom{\psi_L^1}{\psi_L^2}$ has two indices

$$(\Psi_L)^a_\alpha \qquad \begin{array}{c} \alpha = 1,2 \text{ represents the Lorentz index} \\ a = 1,2 \text{ represents the Isospin index} \end{array}$$

while ψ_R has only the Lorentz $\dot{\beta}$ index, and each transformation acts separately on the two indices:

$$\text{Lorentz} \begin{cases} x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ (\Psi'_L)^a_\alpha(x') = (\Lambda_L)_\alpha^{\ \beta} (\Psi_L)^a_\beta(\Lambda^{-1}x') \\ (\psi'_R)^{\dot{\beta}}(x') = (\Lambda_R)^{\dot{\beta}}{}_{\dot{\delta}} (\psi_R)^{\dot{\delta}}(\Lambda^{-1}x'), \end{cases}$$

$$\text{Isospin} \begin{cases} x'^\mu = x^\mu \\ (\Psi'_L)^a_\alpha(x') = U^a_{\ b}(\Psi_L)^b_\alpha(x) \\ (\psi'_R)^{\dot{\beta}}(x') = (\psi_R)^{\dot{\beta}}(x). \end{cases}$$

- Identify the representation of Isospin and Lorentz group the following terms belong to:

  $$\psi_R^\dagger \Psi_L\,, \qquad [(\Psi_L)^a]^\dagger (\sigma^i)^a{}_b \bar{\partial\!\!\!/} \Psi_L^b,$$

  where we have suppressed the Lorentz indices and here $\bar{\partial\!\!\!/} \equiv \bar{\sigma}_\mu \partial^\mu$. Note that in the second term the Pauli matrix is contracted with the Isospin indices while $\bar{\partial\!\!\!/}$ with the Lorentz ones.

Exercise 2: Decomposition of tensor products

SU(2)

If u_a and v_b are doublets of SU(2), decompose into irreducible representations the following tensor products.

- $u_a v_b^* \sim \frac{1}{2} \otimes \overline{\frac{1}{2}}$;
- $u_a v_b \sim \frac{1}{2} \otimes \frac{1}{2}$.

Lorentz

If ψ_α and ϕ_β are left handed spinors and A^μ and B^ν are 4-vectors, decompose into irreducible representations the following tensor products:

- $\psi_\alpha \phi_\beta \sim (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$;
- $A_\mu \psi_\alpha \sim (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0)$;
- $A_\mu B_\nu \sim (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2})$. Focus only on the scalar irrep. Can you guess the others?

Exercise 3: Casimirs of Poincare

In this exercise we will find the Casimirs of the Poincare group, i.e. the operators which commute with all the generators of the group.

We have to search for these Casmirs among the combination of the Poincare generator $J^{\mu\nu}$ and P^μ, and it is clear that these combinations must form a Lorentz scalar (can you prove it?).

Let us start with a generic Lorentz four-vector p^μ and an anti-symmetric tensor $j^{\mu\nu}$. In total they are a collection of 4 and 6 numbers, not operators. Among them we have to find a set of independent Lorentz invariants e_i. By independent we mean that no one of the e_i can be written as function of the others.

A possible strategy to find all the independent invariants is to exploit all the transformation of the Lorentz group in order to set to zero as many components as we can of the two tensors.

For example, if we have only one vector $\vec{v} = (v_x, v_y, v_z)^T$ and our symmetry group is $SO(3)$, then it is clear that exploiting the group transformations we can rotate $\vec{v}$ to $\vec{v}' = (0, 0, v'_z)^T$. This means that there is exactly one rotational invariant quantity we can build out of $\vec{v}$. We can choose this invariant to be $e_v \equiv ||\vec{v}||$, all the others invariants (for instance $||v||^3$) will be function of e_v.

We now follow a similar logic with $j^{\mu\nu}$ and p^μ.

- Write down the transformation of the two tensors under a generic Lorentz transformation Λ^μ_ν.
- Decompose $j^{\mu\nu}$ as

$$j_i^\pm = \frac{j_i \pm i k_i}{2} \,, \qquad \text{where as usual} \qquad k_i = j^{i0} \,, \quad j_i = \frac{\epsilon_{ijk} j^{jk}}{2} \,, \tag{1}$$

  and show how $j_i^\pm$ transform under a Lorentz transformation.
- Show that you can always find a Lorentz transformation such that $j_2^\pm = j_3^\pm = 0$ and write $j^{\mu\nu}$ in this frame.
- Find the residual symmetry group, i.e. the subrugoup of Lorentz which leaves this form of j invariant.
- Show that using this residual group you can set $p_1 = p_2 = 0$.

We have found that in this specific frame we can reduce $j^{\mu\nu}$ and p^μ to only four components, this means that we can build 4 independent invariants. For instance let us consider

$$e_1 = j_{\mu\nu} j^{\mu\nu} \,, \qquad e_2 = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} j_{\mu\nu} j_{\rho\sigma} \,, \qquad e_3 = p^\mu p_\mu \,, \qquad e_4 = w^\mu w_\mu \,, \tag{2}$$

where $w^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} j_{\nu\rho} p_\sigma$.

- Express the four invariants e_i in the specific frame found above, as function of j_1, k_1, p_0, p_3.
- Show that this map is bijective, i.e. that you can also express j_1, k_1, p_0, p_3 as function of the four e_i (you don't need to find the map explicitely).

Now that we have shown the e_i are four independent invariants we can uplift them to operators and look among them for the Casmirs of Poincare.

- Find which of the following operators

$$E_1 = J_{\mu\nu}J^{\mu\nu}\,, \qquad E_2 = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\mu\nu}J_{\rho\sigma}\,, \qquad E_3 = P^\mu P_\mu\,, \qquad E_4 = W^\mu W_\mu\,, \tag{3}$$

  are the two Casimirs of Poincare. In order to do that, you will have to check which two operators satisfy, $[E_i, P^\mu] = [E_i, J^{\mu\nu}] = 0$ (Notice that $[E_i, J^{\mu\nu}] = 0$ is obvious since by construction each E_i is a Lorentz scalar).

Extra:

Up to now you have shown that, given the Poincaré generators P^μ and $J^{\mu\nu}$, one can construct exactly four independent Lorentz scalars $E_i(P, J)$, with $i = 1, \ldots, 4$. You have also identified two of them, which commute with all generators of the Poincaré algebra, so they are Casimir operators.

(a) Explain why any additional Casimir operator $C(P, J)$ must be expressible as a function of these four invariants,

$$C = f(E_1, E_2, E_3, E_4)\,.$$

(b) Show that no nontrivial function $f(E_1, E_2, E_3, E_4)$ that is *independent* of the two Casimirs you already found (E_3, E_4) can commute with all components of P^μ. In other words, prove that if

$$[\, f(E_1, E_2, E_3, E_4),\ P^\mu\,] = 0 \quad \forall\,\mu\,,$$

then f must in fact depend only on E_3 and E_4.

(c) Conclude that the Poincaré algebra in $3 + 1$ dimensions has exactly two independent Casimir operators.