Quantum Field Theory

Homework Set 2

Exercise 1: Scale transformations

The action of a free real massless scalar field in d dimension reads:

$$\mathcal{S} = \int dt d^{d-1}x \; \mathcal{L}(t,x), \qquad \mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi,$$

where $\mu = 0, 1, .., d-1$, $x^0 = t$ and the indices are raised and lowered with the d-dimensional metric $\eta_{\mu\nu} = \text{diag}(1, -1, .., -1)$.
Consider the transformation

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = e^\lambda x^\mu \\ \phi(x) &\longrightarrow \phi'(x') = e^{k\lambda}\phi(e^{-\lambda}x') \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$.

- Compute the value of k such that the above transformation defines a symmetry of the theory.
- For $d = 4$ compute the energy momentum tensor $T^\mu_{\ \nu}$ of the theory. Compute the trace of the energy momentum tensor.
- Consider the *improved energy momentum tensor* $K^\mu_{\ \nu} = T^\mu_{\ \nu} + A\delta^\mu_\nu\Box\phi^2 + B\partial^\mu\partial_\nu\phi^2$. Show, in $d = 4$, that we can choose the values of A and B in such a way that $\partial_\mu K^\mu_{\ \nu} = 0$ and $K^\mu_{\ \mu} = 0$.
- For $d = 4$ compute the Noether's current S^μ associated to the symmetry defined in (1). Express it in terms of the improved energy momentum tensor $K^\mu_{\ \nu}$ and show which constraints $\partial_\mu S^\mu = 0$ imposes on $K^\mu_{\ \nu}$.
- Find for which values of d the addition of the following potentials to the free Lagrangian density doesn't spoil the symmetry (that is to say the transformation is still a symmetry of the new Lagrangian density):

$$\mathcal{L}' = \partial^\mu\phi\partial_\mu\phi - \begin{cases} \frac{m^2}{2!}\phi^2, \\ \text{or} \\ \frac{\beta}{3!}\phi^3, \\ \text{or} \\ \frac{\alpha}{4!}\phi^4. \end{cases}$$

  and compute the dimension (in powers of energy) of the parameters m, β, α for those values of d.
- Discuss this result.

Exercise 2: Charges Algebra

Consider a Lie symmetry group $\mathcal{G}$ described by parameters $\{\alpha^i\}$, acting on coordinates and fields as

$$\begin{aligned} g: &\qquad x^\mu \longrightarrow x'^\mu = x^\mu, \\ g: &\qquad \phi_a(x) \longrightarrow \phi'_a(x') = \mathcal{R}(g)_a{}^b\phi_b(x), \end{aligned}$$

where $\mathcal{R}(g)_a{}^b$ is the representation of an element $g \in \mathcal{G}$.
Show that the Noether's charges Q_i together with the product defined by the Poisson brackets form an Algebra which is isomorphic to the Lie Algebra of $\mathcal{G}$.

Exercise 3: $SO(N)$ and $\mathbb{D}_4$

Part 1. Consider N scalar fields $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x))$ transforming in the fundamental representation of $SO(N)$:

$$\Phi(x) \longrightarrow \Phi'(x) = O\Phi(x), \qquad O \in SO(N).$$

Write the most general Lagrangian $\mathcal{L}$ for $\Phi(x)$ such that

- $\mathcal{L}$ is Lorentz invariant,
- contains only terms with dimension less or equal than four (or equivalently, whose coupling have dimension greater or equal to zero),
- it is invariant under $SO(N)$.

Is the internal symmetry group of $\mathcal{L}$ bigger than $SO(N)$? Consider the case $N = 1$ and $N \geq 2$ separately.

Part 2. Consider now a theory with two scalar fields ϕ_1 and ϕ_2. Build the most general Lorentz invariant Lagrangian with terms up to dimension 4 that is symmetric under the following three transformations separately:

- $\phi_1 \to -\phi_1$
- $\phi_2 \to -\phi_2$
- $\phi_1 \leftrightarrow \phi_2$

What is the difference between this group and $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$? (Reminder: $\mathbb{Z}_2$ is the group formed by two elements $\{1, -1\}$ with their product)
How many elements does this transformation group have?
Can you find a matrix representation for this group?
In which case the Lagrangian you just built is invariant under $O(2)$ as in the first part of this exercise?