Quantum Field Theory

Set 5

Exercise 1: Some Lie Groups

Consider the following Lie groups given here by their defining representations:

$$
\begin{aligned}
&U(N) = \left\{N \times N \text{ complex matrices such that } U^\dagger U = UU^\dagger = 1\right\}, \\
&SU(N) = \left\{N \times N \text{ complex matrices such that } U^\dagger U = UU^\dagger = 1,\ \det U = 1\right\}, \\
&O(N) = \left\{N \times N \text{ real matrices such that } R^T R = R^T R = 1\right\}, \\
&SO(N) = \left\{N \times N \text{ real matrices such that } R^T R = R^T R = 1,\ \det R = 1\right\}, \\
&SL(N, \mathbb{C}) = \{N \times N \text{ complex matrices such that } \det A = 1\ \}.
\end{aligned}
$$

Consider transformations close to the identity and expand a general element of the group as a power series in the Lie parameters

$$D[g(\alpha)] = \mathbb{1} + i\alpha^i T^i + O(\alpha^2).$$

For each of the given groups find the constraints on the T^i and deduce the dimension of the group.

Hint: you will have to use the following property:

$$\det(\mathbb{1} + i\alpha^i T^i + O(\alpha^2)) = 1 + i\alpha^i \mathrm{Tr}(T^i) + O(\alpha^2)$$

Exercise 2: Irreducible representations of $SU(2)$

Consider the Algebra of the group $SU(2)$

$$\left[T^a, T^b\right] = i\epsilon^{abc} T^c.$$

- Find all the commutation rules between the following quantities

$$T^\pm = \frac{T^1 \pm iT^2}{\sqrt{2}}; \qquad T^3.$$

- Show that the *Casimir operator* $J^2 \equiv (T^1)^2 + (T^2)^2 + (T^3)^2$ is proportional to the identity in any irreducible representation: $(\tau)^2 = \mu^2 \times 1_N$. (Here we denote as $(\tau)^2$ the representative of the operator J^2).

Consider an irreducible representation of the Algebra on a given vector space V, endowed with a usual scalar product satisfying $\langle n|n\rangle \equiv ||n\rangle|^2 \geq 0$. Denote $|m\rangle$ an eigenvector of the generator τ^3 relative to the eigenvalue m.

$$\tau^3 |m\rangle = m|m\rangle.$$

- Compute the action of $\tau^\pm$ on $|m\rangle$.
- Prove that $|m| + m^2 \leq \mu^2$.
- Construct the irreducible representation which $|m\rangle$ belongs to.
- Compute the dimension of the representation and show that $\mu^2 = j(j+1)$ with j integer or semi-integer.
- Construct the representation corresponding to $j = 1/2$ and $j = 1$.

Exercise 3: Direct sum and tensor product

Given two finite-dimensional inequivalent irreducible representations D_1 and D_2 of a group G on two vector spaces V_1 and V_2 ($D_1 \in GL(V1)$ and $D_2 \in GL(V_2)$ where $GL(V)$ is the set of all linear transformations from V to itself):

- Show that the operators $D(g) \in GL(V_1 \oplus V_2)$, $g \in G$ defined by

  $$D(g)(v_1 \oplus v_2) \equiv D(g)(v_1, v_2) = (D_1(g)v_1, D_2(g)v_2), \quad v_1 \in V_1,\ v_2 \in V_2, \tag{1}$$

  furnish a (reducible) representation of G (called the direct sum of D_1 and D_2). What is the dimension of the vector space $V_1 \oplus V_2$?

  Given an operator $A \in GL(V_1 \oplus V_2)$ and $A \neq 0$, such that $AD(g) = D(g)A$ for every $g \in G$ show that $A(v_1 \oplus v_2)^T = (\lambda_1 v_1, \lambda_2 v_2)^T$, that is A is a multiple of the identity over both subspaces $(V_1, 0)$ and $(0, V_2)$ of $V_1 \oplus V_2$.

- Show that the operators $D(g) \in GL(V_1 \otimes V_2)$, $g \in G$ defined by

  $$D(g)(v_1 \otimes v_2) = D_1(g)v_1 \otimes D_2(g)v_2, \quad v_1 \in V_1,\ v_2 \in V_2, \tag{2}$$

  where $\otimes$ is associative and distributive (like the usual product), furnish a representation of G (called the direct product of D_1 and D_2). What is the dimension of the vector space $V_1 \otimes V_2$?

  Express the generators of D in terms of the generators of D_1 and D_2. Show that they 'add'.

Exercise 4: Sum of spins in group theoretic language

The statement that two particles have spin 1/2 reflects the fact that each particle can exist in two different states corresponding to the values of the z-component of the angular momentum. These two states define a vector space on which the group $SU(2)$ is represented. Consider the total spin of the bound state of the two particles: this corresponds to taking the tensor product of two $j = 1/2$ representations. Show that the resulting vector space,

$$V = \left\{ |s_1\rangle \otimes |s_2\rangle\,,\ s_i = \pm\frac{1}{2} \right\},$$

can be decomposed in a direct sum of two spaces on which act irreducible representations of $SU(2)$. Find these representations.