

Fermi Normal Coordinates

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

This exercise is dedicated to a particularly interesting coordinate system, namely *Fermi normal coordinates*, which form the mathematical basis of the equivalence principle. Its physical meaning is the following. Suppose that you are freely falling and non-rotating, and that you carry a clock, along with a set of three long rigid rulers which you hold perpendicular to each other. The four coordinates $(X^\alpha) \equiv (\tau, X, Y, Z)$, where τ is the time measured with the clock, and X, Y, Z the lengths measured with the rulers, form your Fermi normal coordinates.

Reference. This exercise is adapted from the book *A Relativist's Toolkit*, by Eric Poisson.

1 The observer's tetrad

Consider an arbitrary spacetime (\mathcal{M}, g) provided with an arbitrary coordinate system $\{x^\mu\}$. Let \mathcal{L} be the worldline of a freely falling observer.

Q1. What can you say about \mathcal{L} ?

Solution Q1

Being the worldline of a freely falling observer, \mathcal{L} is a timelike geodesic.

At a given point $A \in \mathcal{L}$, we consider a *tetrad*, i.e. a set of four four-vectors (e_α) such that

$$g(e_\alpha, e_\beta)|_A = \eta_{\alpha\beta}. \quad (1)$$

Q2. How does one usually call such a set of vectors in geometry?

Solution Q2

The definition of tetrad corresponds to the usual notion of *orthonormal basis*.

Let us further assume that $e_0 = \mathbf{u}$, the observer's four-velocity at A , and let us extend e_α on the whole \mathcal{L} by parallel transporting them from A to any other point $B \in \mathcal{L}$.

Q3. Justify that eq. (1) holds at every point of \mathcal{L} .

Solution Q3

To justify that eq. (1) then holds at every point of \mathcal{L} , we can take the directional derivative of $g(e_\alpha, e_\beta)$ along \mathbf{u} , i.e.,

$$\mathbf{u}[g(e_\alpha, e_\beta)] = g(\nabla_{\mathbf{u}} e_\alpha, e_\beta) + g(e_\alpha, \nabla_{\mathbf{u}} e_\beta) = 0,$$

hence, if $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$ somewhere on \mathcal{L} , it is true everywhere on this curve.

The tetrad field $\{e_\alpha\}$, now defined all along \mathcal{L} , will constitute the axes of the Fermi normal coordinate system centred on the observer.

2 Construction of the coordinates

Let Q be any point in \mathcal{M} . Assume that there is a unique space-like geodesic \mathcal{G} which connects this point to \mathcal{L} orthogonally, in the sense that the (unit) tangent vector \mathbf{v} of \mathcal{G} is orthogonal to $\mathbf{u} = \mathbf{e}_0$ at $P = \mathcal{L} \cap \mathcal{G}$, that is $\mathbf{g}(\mathbf{v}, \mathbf{e}_0)|_P = 0$.

Q4. Show that, at P , there exist three numbers $(v^a)_{a=1,2,3}$ such that $\mathbf{v} = v^a \mathbf{e}_a$.

Solution Q4

At P , the vector \mathbf{v} can be decomposed over the basis $\{\mathbf{e}_\alpha\}$ as $\mathbf{v} = v^\alpha \mathbf{e}_\alpha$. Taking the scalar product with \mathbf{e}_0 , we find

$$0 = \mathbf{g}(\mathbf{v}, \mathbf{e}_0) = v^\alpha \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_0) = -v^0,$$

whence the result.

We now define the geodesic distance s from P to Q as

$$s = \int_P^Q ds = \int_P^Q \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2}$$

along the geodesic \mathcal{G} , where λ can be any parameter along it. The Fermi normal coordinates of Q are then defined as (τ, X^a) , with τ the proper time of P , with respect to an arbitrary origin $O \in \mathcal{L}$ where $\tau = 0$; and with

$$X^a \equiv s v^a. \tag{3}$$

Q5. Summarise this construction with a drawing depicting $\mathcal{L}, Q, \mathcal{G}, P, (\mathbf{e}_\alpha), \mathbf{v}, s, O, \tau$.

Solution Q5

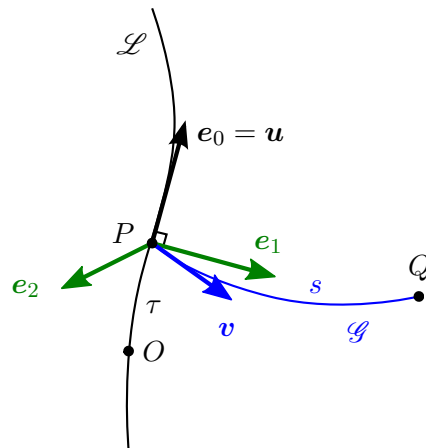


Figure 1 Fermi normal coordinates depicted. As one cannot draw in 4 dimensions, we only represented two ‘spatial’ dimensions and dropped \mathbf{e}_3 .

3 Metric around \mathcal{L}

Suppose that we want to perform the transformation from the arbitrary coordinate system $\{x^\mu\}$ to the Fermi coordinates $\{X^\alpha\}$. This consists of four functions $x^\mu(\tau, X^a = s v^a)$.

Q6. Justify that $\partial x^\mu / \partial \tau|_{X^a=0} = e_0^\mu$ and $\partial x^\mu / \partial s|_{s=0} = v^\mu|_{s=0}$.

Solution Q6

The curves \mathcal{L} and \mathcal{G} are respectively described by $\tau \mapsto x^\mu(\tau, X^a = 0)$ and $s \mapsto x^\mu(\tau, sv^a)$, where in the latter τ and v^a are kept constant. The (normalised) tangent vectors to those curves are therefore obtained by taking the derivative of the aforementioned functions with respect to the parameters τ , s , respectively.

Q7. Deduce that, on \mathcal{L} , we have $\partial x^\mu / \partial X^\alpha = e_a^\mu$. What do you conclude about $g_{\alpha\beta}(\tau, X^a = 0)$?

Solution Q7

For $\alpha = 0$ the relation has already been obtained above,

$$\frac{\partial x^\mu}{\partial X^0} = \frac{\partial x^\mu}{\partial \tau} = e_0^\mu.$$

For $\alpha = a \neq 0$, we just introduce the tetrad decomposition of $\mathbf{v} = v^a \mathbf{e}_a$, so that

$$v^\mu = v^a e_a^\mu = \frac{\partial x^\mu}{\partial s} = \frac{\partial x^\mu}{\partial X^a} \frac{\partial X^a}{\partial s} = v^a \frac{\partial x^\mu}{\partial X^a},$$

which yields $e_a^\mu = \partial x^\mu / \partial X^a$. As a consequence, on \mathcal{L} , we have

$$g_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial X^\alpha} \frac{\partial x^\nu}{\partial X^\beta} = g_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \eta_{\alpha\beta}.$$

The equation implies that the spacetime in the immediate vicinity of any point on the observer's worldline \mathcal{L} appears flat, despite the overall curvature of spacetime due to gravity.

We now wish to determine the first derivatives of the metric on \mathcal{L} .

Q8. Using that the curves defined by $s \mapsto x^\mu(\tau, v^a s)$, for fixed values of τ and v^a , are geodesics, show that $\Gamma^{\alpha}_{bc} = 0$.

Solution Q8

In terms of Fermi coordinates, a curve $s \mapsto x^\mu(\tau, v^a s)$ is just a straight line, with $\tau = \text{cst}$, and $X^a = v^a s$, $v^a = \text{cst}$. Furthermore, since it is by definition a geodesic, we know that it has to satisfy the geodesic equation:

$$0 = \frac{d^2 X^\alpha}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dX^\beta}{ds} \frac{dX^\gamma}{ds} = \Gamma^{\alpha}_{bc} v^b v^c$$

Since this is true for any v^b, v^c , we conclude that $\Gamma^{\alpha}_{bc} = 0$. Interestingly enough, this is true everywhere, and not just on \mathcal{L} .

Q9. Demonstrate that the remaining coefficients, namely $\Gamma^{\alpha}_{\tau\tau}$ and $\Gamma^{\alpha}_{\tau b}$ vanish because of the parallel transport condition on the observer's tetrad. Deduce that

$$\partial_\alpha g_{\beta\gamma}(\tau, X^a = 0) = 0. \quad (4)$$

Solution Q9

First let's determine the Fermi components of the tetrad vectors \mathbf{e}_β . Just like for any coordinate transformation we can transform the tetrad components from the coordinate system $\{x^\mu\}$ to the Fermi Normal Coordinates on \mathcal{L}

$$e_\beta^\mu = \frac{\partial x^\mu}{\partial X^\alpha} e_\beta^\alpha = e_\alpha^\mu e_\beta^\alpha$$

Thus $e_\beta^\alpha = \delta_\beta^\alpha$, which is the consequence of the orthonormality of the tetrad.

Adding to this the parallel transport condition on e_β , we get

$$\nabla_{\mathbf{u}} e_\beta = \frac{de_\beta^\alpha}{d\tau} + \Gamma^\alpha_{\tau\gamma} e_\beta^\gamma = \Gamma^\alpha_{\tau\beta} = 0$$

all along \mathcal{L} , which sets to zero the last Christoffel symbols (note that $\frac{de_\beta^\alpha}{d\tau} = 0$ as $e_\beta^\alpha = \delta_\beta^\alpha = \text{cst}$). The “fully down” versions of the Christoffel symbols $\Gamma_{\alpha\beta\gamma}$ also vanish along this worldline, because the indices are raised and lowered by $\eta^{\alpha\beta}, \eta_{\alpha\beta}$, which merely adds a minus sign for $\alpha = 0$. Since the first derivatives of the metric are linear combinations of the $\Gamma_{\alpha\beta\gamma}$, they also vanish on \mathcal{L} .

Q10. Show that, on \mathcal{L} , $g_{\alpha\beta,\gamma\tau} = 0$ and $g_{\tau\tau,ab} = -2R_{\tau a\tau b}$.

Solution Q10

The first derivative $g_{\alpha\beta,\gamma}$ are zero all along \mathcal{L} so $g_{\alpha\beta,\gamma\tau} = 0$. Regarding $g_{\tau\tau,ab}$, we start from the definition of the Riemann tensor and simplify it using the previous results: still on \mathcal{L} , we have

$$\begin{aligned} R_{\tau a\tau b} &= g_{\tau\gamma} \left(\Gamma^\gamma_{ab,\tau} - \Gamma^\gamma_{a\tau,b} + \Gamma^\gamma_{\delta\tau} \Gamma^\delta_{ab} - \Gamma^\gamma_{\delta b} \Gamma^\delta_{a\tau} \right) \\ &= \Gamma_{\tau ab,\tau} - \Gamma_{\tau a\tau,b} \quad (\text{since } \Gamma^\alpha_{\beta\gamma} = 0) \\ &= \frac{1}{2} [g_{\tau a,b\tau} + g_{\tau b,a\tau} - g_{ab,\tau\tau}] - \frac{1}{2} [g_{\tau a,\tau b} + g_{\tau\tau,ab} - g_{a\tau,\tau b}] \\ &= -\frac{1}{2} g_{\tau\tau,ab} \end{aligned}$$

as this term is the only one which does not contain a derivative with respect to τ .

Proof of : $\Gamma_{\tau ab,\tau} = \frac{1}{2} [g_{\tau a,b\tau} + g_{\tau b,a\tau} - g_{ab,\tau\tau}]$

$$\begin{aligned} \Gamma_{\tau ab} &= g_{\tau\rho} \Gamma_{ab}^\rho \\ &= g_{\tau\rho} \left(\frac{1}{2} g^{\rho\sigma} (\partial_a g_{b\sigma} + \partial_b g_{a\sigma} - \partial_\sigma g_{ab}) \right) \\ &= \frac{1}{2} (\partial_a g_{\tau b} + \partial_b g_{\tau a} - \partial_\tau g_{ab}) \end{aligned}$$

Therefore

$$\Gamma_{\tau ab,\tau} = \frac{1}{2} (g_{\tau b,a\tau} + g_{\tau a,b\tau} - g_{ab,\tau\tau})$$

It is possible to show that the other second derivatives of the metric on \mathcal{L} can be expressed in terms of the Riemann tensor. However, the calculations are slightly more involved, so that you can *accept* the following relations without proving them:

$$g_{\tau a,bc} = -\frac{2}{3} (R_{\tau bac} + R_{\tau cab}), \quad (5)$$

$$g_{ab,cd} = -\frac{1}{3} (R_{acbd} + R_{adbc}). \quad (6)$$

Q11. Conclude by giving the expression of the metric, in terms of Fermi coordinates around \mathcal{L} , up to second order in X^a .

Solution Q11

Expanding the metric at second order around \mathcal{L} , it is straightforward to obtain

$$ds^2 = -(1 + R_{\tau a \tau b} X^a X^b) d\tau^2 - \frac{4}{3} R_{\tau b a c} X^b X^c d\tau dX^a + \left[\delta_{ab} - \frac{1}{3} R_{acbd} X^c X^d \right] dX^a dX^b + \mathcal{O}(s^3).$$

Proof:

We perform a Taylor expansion of the metric at point P :

$$g_{\mu\nu}(X) = g_{\mu\nu}(P) + \left(\frac{\partial g_{\mu\nu}}{\partial X^\alpha} \right)_P X^\alpha + \frac{1}{2} \left(\frac{\partial^2 g_{\mu\nu}}{\partial X^\alpha \partial X^\beta} \right)_P X^\alpha X^\beta + \mathcal{O}(X^3)$$

Next, we proceed to examine three distinct scenarios based on varying combinations of indices (remember that the first derivative of the metric always vanishes, and any second derivative of the metric, when taken with respect to proper time, similarly results in zero)

CASE I. $\mu = \nu = \tau$:

$$\begin{aligned} g_{\tau\tau}(X) &= g_{\tau\tau}(P) + \left(\frac{\partial g_{\tau\tau}}{\partial X^\alpha} \right)_P X^\alpha + \frac{1}{2} \left(\frac{\partial^2 g_{\tau\tau}}{\partial X^\alpha \partial X^\beta} \right)_P X^\alpha X^\beta + \mathcal{O}(X^3) \\ &= \eta_{\tau\tau}(P) + \frac{1}{2} g_{\tau\tau,ab} X^a X^b \quad , \quad (\alpha = a \quad , \beta = b) \\ &= -1 - R_{\tau a \tau b} X^a X^b \end{aligned}$$

CASE II. $\mu = \tau, \nu = a$:

$$\begin{aligned} g_{\tau a}(X) &= g_{\tau a}(P) + \left(\frac{\partial g_{\tau a}}{\partial X^\alpha} \right)_P X^\alpha + \frac{1}{2} \left(\frac{\partial^2 g_{\tau a}}{\partial X^\alpha \partial X^\beta} \right)_P X^\alpha X^\beta + \mathcal{O}(X^3) \\ &= \eta_{\tau a}(P) + \frac{1}{2} g_{\tau a,bc} X^b X^c \quad , \quad (\alpha = b \quad , \beta = c) \\ &= \frac{1}{2} \left(-\frac{2}{3} (R_{\tau b a c} + R_{\tau c a b}) \right) X^b X^c \\ &= -\frac{2}{3} R_{\tau b a c} X^b X^c \quad , \quad (\text{using the symmetry properties of Riemann tensor}) \end{aligned}$$

The same applies to the case: $\mu = a, \nu = \tau \longrightarrow g_{a\tau} = -\frac{2}{3} R_{\tau b a c}$.

CASE III. $\mu = a, \nu = b$:

$$\begin{aligned} g_{ab}(X) &= g_{ab}(P) + \left(\frac{\partial g_{ab}}{\partial X^\alpha} \right)_P X^\alpha + \frac{1}{2} \left(\frac{\partial^2 g_{ab}}{\partial X^\alpha \partial X^\beta} \right)_P X^\alpha X^\beta + \mathcal{O}(X^3) \\ &= \eta_{ab}(P) + \frac{1}{2} g_{ab,cd} X^c X^d \quad , \quad (\alpha = c \quad , \beta = d) \\ &= \delta_{ab} + \frac{1}{2} \left(-\frac{1}{3} (R_{acbd} + R_{adbc}) \right) X^c X^d \\ &= \delta_{ab} - \frac{1}{3} R_{acbd} X^c X^d \quad , \quad (\text{using the symmetry properties of Riemann tensor}) \end{aligned}$$

Substituting these results into the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, we derive the desired outcome.