Nordström's Theory of Gravity

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Gunnar Nordström

Proposed in 1912, that is, only three years before Einstein's general relativity (GR), Nordström's theory of gravity is probably the first consistent relativistic theory of gravitation ever proposed. Its geometric reformulation by Einstein and Fokker in 1914 definitely paved the way towards GR, which shows that the birth of the latter has been the fruit of a collective research effort. Although it turned out to be an incorrect description of nature, Nordström's theory still has an interesting pedagogical interest, in particular for the understanding of some extensions of GR, known as scalar-tensor theories.

1 Original formulation

In its original form, Nordström's theory is technically very close to any relativistic field theory, like electrodynamics. The idea consists in considering Newton's gravitational field Φ as a scalar field in Minkowski spacetime, conformally coupled to matter. If we assume that matter is made of N massive point-particles, then the action of the system reads

$$S = -\frac{1}{8\pi G}\int \mathrm{d}^4x\ \eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \sum_{a=1}^{N} S_a[\boldsymbol{y}_a, \Phi], \tag{1}$$

where $\{x^\mu\}$ is an inertial coordinate system, $\eta_{\mu\nu}$ is the Minkowski metric, and S_a is the action of a point particle, with a slight modification:

$$S_a[\boldsymbol{y}_a, \Phi] = -m_a \int \mathrm{d}\tau_a\ [1 + \Phi(\boldsymbol{y}_a)]. \tag{2}$$

In the above, $\boldsymbol{y}_a$ denote the trajectory of particle a, τ_a its proper time, m_a its mass, and $(1+\Phi)$, called conformal factor, will be responsible for the coupling between matter and gravitation[1].

Q1. Using a similar trick as last week, show that the action of a single particle can be rewritten

$$S_a = -\int \mathrm{d}^4x\ (1+\Phi)\,\gamma m_a(1-v^2)\delta_{\mathrm{D}}[x^i - y_a^i(t)], \tag{3}$$

where v is the velocity of the particle, and γ the corresponding Lorentz factor. Conclude that the total matter action reads

$$\sum_{a=1}^{N} S_a[\boldsymbol{y}_a, \Phi] = -\int \mathrm{d}^4x\ (1+\Phi)(\rho - 3P), \tag{4}$$

[1] This should remind you of electrodynamics, where charged particles have a coupling of this form, Φ being replaced by $qu^\mu A_\mu$, q being the charge of the particle and $\boldsymbol{u}$ its four-velocity.

where you will give the expression and physical interpretation of ρ and P.

Solution Q1

Starting from a single particle, and dropping indices a for simplicity, we transform its action into a four-dimensional integral as

$$\begin{aligned} S[\boldsymbol{y}, \Phi] &= -m \int \mathrm{d}\tau \int \mathrm{d}^4x \; \delta_{\mathrm{D}}^{(4)}[\boldsymbol{x} - \boldsymbol{y}(\tau)][1 + \Phi(\boldsymbol{x})] \\ &= -m \int \mathrm{d}^4x \; (1 + \Phi(\boldsymbol{x})) \int \mathrm{d}\tau \; \delta_{\mathrm{D}}^{(4)}[\boldsymbol{x} - \boldsymbol{y}(\tau)] \\ &= -m \int \mathrm{d}^4x \; (1 + \Phi(\boldsymbol{x})) \int \mathrm{d}\tau \; \delta(t - y^0(\tau)) \cdot \delta^{(3)}[x^i - y^i(\tau)] \end{aligned} \tag{5}$$

The delta function $\delta(t - y^0(\tau))$is nonzero only when $t = y^0(\tau)$. Let τ_t be the proper time at which this condition is satisfied. Around τ_t, we can approximate $y^0(\tau)$ as

$$y^0(\tau) \approx \underbrace{y^0(\tau_t)}_{t} + (\mathrm{d}y^0/\mathrm{d}\tau)_{\tau_t}(\tau - \tau_t)$$

The delta function then becomes:

$$\int \mathrm{d}\tau \; \delta(t - y^0(\tau)) = \int \mathrm{d}\tau \delta((\mathrm{d}y^0/\mathrm{d}\tau)_{\tau_t}(\tau - \tau_t)) = \int \mathrm{d}\tau \frac{1}{|(\mathrm{d}y^0/\mathrm{d}\tau)_{\tau_t}|} \delta(\tau - \tau_t) = \frac{1}{\gamma}$$

where $\gamma = (\mathrm{d}y^0/\mathrm{d}\tau)_{\tau_t}$ is the Lorentz factor of the particle. Substituting this into (5) we get

$$S[\boldsymbol{y}, \Phi] = -\int \mathrm{d}^4x \; (1 + \Phi) \, \frac{m}{\gamma} \, \delta_{\mathrm{D}}^{(3)}[x^i - y^i(t)],$$

We can slightly rewrite $1/\gamma$ as

$$\frac{1}{\gamma} = \gamma(1 - v^2)$$

which returns the desired result. Summing all the actions then yields

$$\begin{aligned} \sum_{a=1}^{N} S_a[\boldsymbol{y}_a, \Phi] &= -\int \mathrm{d}^4x \, (1 + \Phi) \sum_{a=1}^{N} \gamma_a m_a (1 - v_a^2) \delta_{\mathrm{D}}[x^i - y_a^i(t)] \\ &= -\int \mathrm{d}^4x \; (1 + \Phi)(\rho - 3P) \end{aligned}$$

where we introduced

$$\begin{aligned} \rho &\equiv \sum_{a=1}^{N} \gamma_a m_a \, \delta_{\mathrm{D}}[x^i - y_a^i(t)], \\ P &\equiv \sum_{a=1}^{N} \frac{\gamma_a m_a v_a^2}{3} \, \delta_{\mathrm{D}}[x^i - y_a^i(t)], \end{aligned}$$

which represent, respectively, the energy density and kinetic pressure fields of the system of N particles.

Q2. Taking the derivative of the total action S with respect to Φ, show that the associated field equation reads

$$\Box \Phi = 4\pi G(\rho - 3P) \; , \tag{6}$$

with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. In which limit is it equivalent to the Poisson equation of Newtonian gravity? Which new phenomena can it describe?

Solution Q2

For a small variation $\delta\Phi$ of the scalar field, the total action varies as

$$\delta S = -\frac{1}{8\pi G}\int \mathrm{d}^4x\, \eta^{\mu\nu}\left(\partial_\mu\delta\Phi\,\partial_\nu\Phi + \partial_\mu\Phi\,\partial_\nu\delta\Phi\right) + \sum_{a=1}^{N}\delta S_a[\boldsymbol{y}_a, \Phi],$$

Using the symmetrical properties of $\eta^{\mu\nu}$ and the expression for matter action described in the equation (4) we can express the variation of the action as

$$\delta S = -\frac{1}{4\pi G}\int \mathrm{d}^4x\, \eta^{\mu\nu}\partial_\mu\delta\Phi\,\partial_\nu\Phi - \int \mathrm{d}^4x\,\delta\Phi\,(\rho - 3P) \tag{7}$$

The first integral can be calculated using integration by parts. For our case, we choose

$$u = \partial_\nu\Phi \quad , \quad dv = \eta^{\mu\nu}\partial_\mu\delta\Phi\,\mathrm{d}^4x,$$

We then compute du and v

$$du = \partial_\mu\partial_\nu\Phi\,\mathrm{d}x^\mu \quad , \quad v = \eta^{\mu\nu}\delta\Phi$$

Now applying the integration by parts formula

$$\int u\,dv = uv - \int v\,du,$$

we can write

$$\int \mathrm{d}^4x\,\eta^{\mu\nu}\partial_\mu\delta\Phi\,\partial_\nu\Phi = \eta^{\mu\nu}\delta\Phi\,\partial_\nu\Phi\,|_{\text{boundary}} - \int \mathrm{d}^4x\,\eta^{\mu\nu}\delta\Phi\,\partial_\mu\partial_\nu\Phi.$$

The first term on the right side is a boundary term. If we assume that the variation $\delta\Phi$ vanishes at the boundaries of the integration domain, this boundary term drops out. Inserting this result into equation (7) and factoring out $\delta\Phi$ we get

$$\delta S = \int \mathrm{d}^4x\,\delta\Phi\left[\frac{\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi}{4\pi G} - (\rho - 3P)\right]$$

Finally, the field equation is obtained by setting $\delta S = 0$, resulting in

$$\Box\Phi = 4\pi G(\rho - 3P)$$

In the limit of non-relativistic matter particles, for which $v \ll 1$ and hence $P \ll \rho$, and if we assume that the gravitational field is slowly varying ($\partial_t\Phi \ll |\vec{\partial}\Phi|$), then we recover the Poisson equation

$$\Delta\Phi = 4\pi G\rho \tag{8}$$

of Newtonian gravity. However, in general, Nordström's equation describes phenomena that do not exist in Newtonian physics:

- There exist propagating solutions, similar to *gravitational waves*, in vacuum, since d'Alembert's equation $\Box\Phi$ has such solutions.
- Like in electromagnetism, the solutions of Nordström's equations are *retarded potentials*, instead of the Newtonian's instantaneous potential. Gravitational information propagates at the speed of light in Nordström's theory.
- Not only rest mass ρ is a source for Φ, but also kinetic energy via pressure P. In Nordström's theory, and contrary to GR, a cold gas is heavier than a hot gas, and a gas of photons ($v = 1$ so $P = \rho/3$) does not gravitate.

These new elements are qualitatively comparable to GR, but turn out to fail in describing experiments and observations in the details, contrary to GR.

The above shows how matter generates a gravitational field. Let us now see how this field can, in turn, affect the motion of matter. For that purpose, let us consider only one of the particle described by the action (2). We will drop the subscripts a to alleviate notation. If the trajectory of the particle is parametrised by an arbitrary λ, this action thus reads

$$S_1[\boldsymbol{x}] = -m \int \mathrm{d}\lambda \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \, [1 + \Phi(\boldsymbol{x}(\lambda))]. \tag{9}$$

Q3. Show that equation of motion of the particle dictated by the action (9) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau} [(1 + \Phi) u^\mu] = -\partial^\mu \Phi \tag{10}$$

Solution Q3

We consider the variation of the action S_1 for a small shift $\delta x^\mu(\lambda)$ in the trajectory of the particle,

$$\delta S_1 = -m \left[\int \mathrm{d}\lambda \, (1 + \Phi) \delta \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} + \int \mathrm{d}\tau \, \delta\Phi \right]. \tag{11}$$

Starting from the first term in the right side of the equation (11) we can write

$$\begin{aligned}
\int \mathrm{d}\lambda \, (1 + \Phi) \delta \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} &= - \int \mathrm{d}\lambda \, (1 + \Phi) \left(-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \right)^{-1/2} \eta_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda} \\
&= - \int \mathrm{d}\tau \, (1 + \Phi) \, \eta_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau} \\
&= \int \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\tau} [(1 + \Phi) u_\mu] \, \delta x^\mu,
\end{aligned}$$

where in the second line we changed the integration variable to proper time using the chain rule

$$d\tau = \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda \quad \text{and} \quad \frac{d}{d\lambda} = \frac{d\tau}{d\lambda} \frac{d}{d\tau} = \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \frac{d}{d\tau}$$

and in the last step we performed an integration by parts by choosing

$$u = (1 + \Phi) \eta_{\mu\nu} u^\nu \quad , \quad dv = \frac{d}{d\tau} (\delta x^\mu) d\tau$$

$$du = \frac{d}{d\tau} ((1 + \Phi) \eta_{\mu\nu} u^\nu) d\tau \quad , \quad v = \delta x^\mu$$

and subsequently

$$- \int \mathrm{d}\tau \, (1 + \Phi) \, \eta_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau} = \underbrace{- (1 + \Phi) \eta_{\mu\nu} u^\nu \delta x^\mu |_{\text{boundary}}}_{0} + \int \mathrm{d}\tau \, \frac{d}{\mathrm{d}\tau} ((1 + \Phi) \underbrace{\eta_{\mu\nu} u^\nu}_{u_\mu}) \, \delta x^\mu$$

where again the boundary term drops out because δx^μ is zero at the endpoints.

On the other hand, the second term on the right side of the equation (11) reads

$$\int \mathrm{d}\tau \, \delta\Phi = \int \mathrm{d}\tau \, \partial_\mu \Phi \, \delta x^\mu$$

and by setting $\delta S_1 = 0$ we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(1+\Phi)u^\mu\right] = -\partial^\mu \Phi$$

as required. Note that this could also be written

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = -\frac{1}{(1+\Phi)}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}u^\mu + \partial^\mu\Phi\right) = -(u^\mu u^\nu + \eta^{\mu\nu})\partial_\nu \ln(1+\Phi),$$

since $\mathrm{d}/\mathrm{d}\tau = u^\nu \partial_\nu$. The tensor $\perp_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu$ can be seen as the projector onto the hyperplane orthogonal to $\boldsymbol{u}$ i.e. space for the particle. In other words, the gravitational force felt by the particle is purely spatial; this actually ensures that the four-velocity remains normalised $\eta_{\mu\nu}u^\mu u^\nu = -1$.

Q4. In which regime do we recover Newton's second law?

Solution Q4

In the non-relativistic regime ($v \ll 1$) we can write $\tau \approx t$ and for weak values of the field $\Phi \ll 1$, we have, at lowest order,

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial^i \Phi,$$

which is Newton's second law. Typically, with gravitational experiments on Earth it is practically impossible to distinguish between Newton, Nordström, or Einstein.

Q5. Consider the following situation: there exists a frame in which Φ is homogeneous in space, but evolving with time $\Phi(t)$. What is the corresponding gravitational force? Does this happen in Newtonian gravity?

Solution Q5

We assume that the coordinate system $\{x^\mu\} = \{t, x^i\}$ is adapted to the homogeneity of the field, so that $\Phi = \Phi(t)$ and hence $\partial_i \Phi = 0$. In general, the particle is moving with respect to this particular homogeneity frame with velocity v^i, and its four-velocity thus reads $u^\mu = \gamma(1, v^i)$. The equation of motion then becomes

$$\frac{\mathrm{d}(1+\Phi)\gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}\Phi}{\mathrm{d}t}$$
$$\frac{\mathrm{d}(1+\Phi)\gamma v^i}{\mathrm{d}\tau} = 0.$$

which can be combined to yield (with $\mathrm{d}t/\mathrm{d}\tau = \gamma$)

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{1}{\gamma^2}\frac{\mathrm{d}\ln(1+\Phi)}{\mathrm{d}t}v^i.$$

Proof: Starting from the second equation, we can use the chain rule $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt}$ and then divide the whole equation by γ, and further expand the equation

$$\begin{aligned}\frac{\mathrm{d}(1+\Phi)\gamma v^i}{\mathrm{d}\tau} = \frac{\mathrm{d}(1+\Phi)\gamma v^i}{\mathrm{d}t} &= 0 \\ = (1+\Phi)\gamma\frac{dv^i}{dt} + v^i\frac{d}{dt}((1+\Phi)\gamma) &= 0\end{aligned} \tag{12}$$

On the other hand we can also use the chain rule for the first equation

$$\frac{\mathrm{d}(1+\Phi)\gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}(1+\Phi)\gamma}{\mathrm{d}t} = \frac{1}{\gamma}\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

Substituting this into the equation (12), we get

$$(1+\Phi)\gamma\frac{dv^i}{dt} + v^i\frac{1}{\gamma}\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0$$

or more simply:

$$\frac{dv^i}{dt} = -\frac{1}{\gamma^2(1+\Phi)}\frac{\mathrm{d}\Phi}{\mathrm{d}t}v^i$$

which can be also formulated as

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\frac{1}{\gamma^2}\frac{\mathrm{d}\ln(1+\Phi)}{\mathrm{d}t}\,v^i.$$

Thus, the gravitational field exerts a sort of friction on the particle, with a coefficient proportional to $\dot{\Phi}$. If for some reason the potential decreases, then the friction is actually an anti-friction, which accelerates the particle.

This does not happen in Newtonian gravity, where the gravitational force is related to the gradient of the potential only. The main reason for this difference is that, in Newtonian gravity, the mass is inalterable, in particular it does not depend on the gravitational potential, contrary to what we find here.

2 The Einstein-Fokker formulation

In 1914, Einstein and Fokker found that Nordström's gravity could be reformulated in terms of a curved space-time. This was a dramatic change of paradigm: instead of viewing gravity as a force, mediated by a field in Minkowski space-time, as in Nordström's original formulation, it started to be seen as a distortion of the geometry of space and time.

Consider the metric defined by

$$g_{\mu\nu} = (1+\Phi)^2\eta_{\mu\nu}, \tag{13}$$

where $\{x^\mu\}$ is still assumed to be an inertial coordinate system with respect to $\eta_{\mu\nu}$, for simplicity.[2]

Q6. Show that the Christoffel symbols of $g_{\mu\nu}$ are

$$\Gamma^\rho{}_{\mu\nu} = \left(\delta^\rho_\mu\partial_\nu + \delta^\rho_\nu\partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\partial_\sigma\right)\ln(1+\Phi). \tag{14}$$

Solution Q6

This follows immediately from the definition of the Christoffel symbols

$$\begin{aligned}\Gamma^\rho{}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\rho g_{\mu\nu}\right)\\ &= \frac{1}{2}(1+\Phi)^{-2}\eta^{\rho\sigma}\left[2(1+\Phi)\partial_\nu\Phi\,\eta_{\sigma\mu} + 2(1+\Phi)\partial_\mu\Phi\,\eta_{\sigma\nu} - 2(1+\Phi)\partial_\sigma\Phi\,\eta_{\mu\nu}\right]\\ &= \left(\delta^\rho_\mu\partial_\nu + \delta^\rho_\nu\partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\partial_\sigma\right)\ln(1+\Phi).\end{aligned}$$

[2] A more general formulation would be to write $\boldsymbol{g} = (1+\Phi)^2\boldsymbol{f}$, where $\boldsymbol{f}$ would denote the flat Minkowski metric, but we do not need such a generality here.

Q7. From the above, show that the Ricci scalar reads $R = -6(1+\Phi)^{-3}\Box\Phi$, with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Solution Q7

Beginning with the definition of the Riemann tensor, we have

$$R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\sigma\lambda}\Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu}$$

To obtain the Ricci tensor from here, we proceed to contract the first and third indices of the Riemann tensor

$$R_{\mu\nu} = g^{\sigma\rho} R_{\rho\mu\sigma\nu}$$

where

$$R_{\rho\mu\sigma\nu} = g_{\rho\lambda} R^\lambda{}_{\mu\sigma\nu}$$

This allows us to rewrite the Ricci tensor as

$$R_{\mu\nu} = g^{\sigma\rho} g_{\rho\lambda} R^\lambda{}_{\mu\sigma\nu} = \delta^\sigma_\lambda R^\lambda{}_{\mu\sigma\nu} = R^\lambda{}_{\mu\lambda\nu}$$

Taking the contraction process a step further, we find the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\lambda{}_{\mu\lambda\nu} = g^{\mu\nu} R^\rho{}_{\mu\rho\nu}$$

In the final step, we have simply relabeled the dummy index λ as ρ. Finally we have

$$\begin{aligned} R &= g^{\mu\nu}\left(\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\rho}\right) \\ &= (1+\Phi)^{-2}\eta^{\mu\nu}\left(\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\rho}\right) \end{aligned}$$

Each term, individually, yields

$$\begin{aligned} \eta^{\mu\nu}\partial_\rho \Gamma^\rho{}_{\mu\nu} &= -2\Box \ln(1+\Phi) \\ \eta^{\mu\nu}\partial_\nu \Gamma^\rho{}_{\mu\rho} &= 4\Box \ln(1+\Phi) \\ \eta^{\mu\nu}\Gamma^\rho{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu} &= -8\,\frac{(\partial\Phi)^2}{(1+\Phi)^2} \\ \eta^{\mu\nu}\Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\rho} &= -2\frac{(\partial\Phi)^2}{(1+\Phi)^2} \end{aligned} \tag{15}$$

which, using $\Box \ln(1+\Phi) = (1+\Phi)^{-1}\Box\Phi - (1+\Phi)^{-2}(\partial\Phi)^2$, gives the desired result.

Here we prove the first and third lines of (15), as the others follow the same logic.

Proof of : $\eta^{\mu\nu}\partial_\rho \Gamma^\rho{}_{\mu\nu} = -2\Box \ln(1+\Phi)$

Using the equation (14), it follows immediately

$$\begin{aligned} \eta^{\mu\nu}\partial_\rho \Gamma^\rho{}_{\mu\nu} &= \eta^{\mu\nu}\partial_\rho\left(\delta^\rho_\mu \partial_\nu + \delta^\rho_\nu \partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\partial_\sigma\right)\ln(1+\Phi) \\ &= \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \eta^{\mu\nu}\partial_\nu\partial_\mu - 4\eta^{\rho\sigma}\partial_\rho\partial_\sigma\right)\ln(1+\Phi) \\ &= \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \eta^{\mu\nu}\partial_\nu\partial_\mu - 4\eta^{\mu\nu}\partial_\mu\partial_\nu\right)\ln(1+\Phi) \\ &= -2\ \eta^{\mu\nu}\partial_\mu\partial_\nu \ln(1+\Phi) \\ &= -2\Box \ln(1+\Phi) \end{aligned}$$

Proof of : $\eta^{\mu\nu}\Gamma^\rho{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu} = -8\,\frac{(\partial\Phi)^2}{(1+\Phi)^2}$

First, let's find ${\Gamma^\rho}_{\sigma\rho}$

$$\begin{aligned}{\Gamma^\rho}_{\sigma\rho} &= \left(\delta^\rho_\sigma\partial_\rho + \delta^\rho_\rho\partial_\sigma - \eta_{\sigma\rho}\eta^{\rho\lambda}\partial_\lambda\right)\ln(1+\Phi)\\ &= \left(\partial_\sigma + 4\partial_\sigma - \eta_{\sigma\rho}\eta^{\rho\lambda}\partial_\lambda\right)\ln(1+\Phi)\\ &= \left(5\partial_\sigma - \delta^\lambda_\sigma\partial_\lambda\right)\ln(1+\Phi)\\ &= 4\partial_\sigma\ln(1+\Phi)\end{aligned}$$

We also have ${\Gamma^\sigma}_{\mu\nu}$ as:

$${\Gamma^\sigma}_{\mu\nu} = \left(\delta^\sigma_\mu\partial_\nu + \delta^\sigma_\nu\partial_\mu - \eta_{\mu\nu}\eta^{\sigma\lambda}\partial_\lambda\right)\ln(1+\Phi)$$

Next, multiply $\eta^{\mu\nu}{\Gamma^\rho}_{\sigma\rho}{\Gamma^\sigma}_{\mu\nu}$:

$$\begin{aligned}\eta^{\mu\nu}{\Gamma^\rho}_{\sigma\rho}{\Gamma^\sigma}_{\mu\nu} &= \eta^{\mu\nu}\cdot 4\partial_\sigma\ln(1+\Phi)\cdot\left(\delta^\sigma_\mu\partial_\nu + \delta^\sigma_\nu\partial_\mu - \eta_{\mu\nu}\eta^{\sigma\lambda}\partial_\lambda\right)\ln(1+\Phi)\\ &= 4\eta^{\mu\nu}\Big(\partial_\mu\ln(1+\Phi)\partial_\nu\ln(1+\Phi) + \partial_\nu\ln(1+\Phi)\partial_\mu\ln(1+\Phi)\\ &\qquad - \eta_{\mu\nu}\eta^{\sigma\lambda}\partial_\sigma\ln(1+\Phi)\partial_\lambda\ln(1+\Phi)\Big)\\ &= 8\partial^\mu\ln(1+\Phi)\partial_\mu\ln(1+\Phi) - 16\eta^{\sigma\lambda}\partial_\sigma\ln(1+\Phi)\partial_\lambda\ln(1+\Phi)\\ &= 8\partial^\mu\ln(1+\Phi)\partial_\mu\ln(1+\Phi) - 16\partial^\lambda\ln(1+\Phi)\partial_\lambda\ln(1+\Phi)\\ &= -8\partial^\mu\ln(1+\Phi)\partial_\mu\ln(1+\Phi)\\ &= -8\frac{(\partial\Phi)^2}{(1+\Phi)^2}\end{aligned}$$

In the last steps, we used the fact that $\partial^\mu\ln(1+\Phi) = \frac{\partial^\mu\Phi}{1+\Phi}$ and $(\partial\Phi)^2 = -\partial^\mu\Phi\partial_\mu\Phi$

Therefore we have

$$R = (1+\Phi)^{-2}\left(-6\Box\ln(1+\Phi) - 6\frac{(\partial\Phi)^2}{(1+\Phi)^2}\right)$$

which, using $\Box\ln(1+\Phi) = (1+\Phi)^{-1}\Box\Phi - (1+\Phi)^{-2}(\partial\Phi)^2$

Proof of : $\Box\ln(1+\Phi) = (1+\Phi)^{-1}\Box\Phi - (1+\Phi)^{-2}(\partial\Phi)^2$

$$\begin{aligned}\Box\ln(1+\Phi) &= \eta^{\mu\nu}\partial_\nu\partial_\mu\ln(1+\Phi)\\ &= \eta^{\mu\nu}\partial_\nu\left(\frac{1}{1+\Phi}\partial_\mu\Phi\right)\\ &= \eta^{\mu\nu}\left(-\frac{1}{(1+\Phi)^2}\partial_\nu\Phi\partial_\mu\Phi + \frac{1}{1+\Phi}\partial_\nu\partial_\mu\Phi\right)\\ &= -\frac{1}{(1+\Phi)^2}\eta^{\mu\nu}\partial_\nu\Phi\partial_\mu\Phi + \frac{1}{1+\Phi}\eta^{\mu\nu}\partial_\nu\partial_\mu\Phi\\ &= -\frac{1}{(1+\Phi)^2}(\partial\Phi)^2 + \frac{1}{1+\Phi}\Box\Phi\end{aligned}$$

gives the desired result

$$R = (1+\Phi)^{-2}\left(-6(1+\Phi)^{-1}\Box\Phi + \cancel{6(1+\Phi)^{-2}(\partial\Phi)^2} - \cancel{6\frac{(\partial\Phi)^2}{(1+\Phi)^2}}\right) = -6(1+\Phi)^{-3}\Box\Phi$$

Q8. Demonstrate that the action (9) for a point particle is equivalent to its general-relativistic counterpart, with the metric $\boldsymbol{g}$. What can you say about free fall in Nordström's theory?

Solution Q8

The general-relativistic action for a point particle being

$$S_1[\boldsymbol{x}] = -m \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} = -m \int \mathrm{d}\lambda \, |1+\Phi| \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}},$$

we indeed recover the Nordström case (9), assuming that $1+\Phi \geq 0$. Therefore, being freely falling in Nordström's theory is equivalent to following time-like geodesics in a curved space-time with metric $g_{\mu\nu} = (1+\Phi)^2 \eta_{\mu\nu}$.

Q9. Using the results of a previous exercise sheet, give the expression of the corresponding stress-energy tensor, and conclude that the quantity $\rho - 3P$ introduced in eq. (6) reads

$$\rho - 3P = -(1+\Phi)^3 T, \tag{16}$$

where $T = g_{\mu\nu} T^{\mu\nu}$ is the trace of the total stress-energy tensor of the system of N particles.

Solution Q9

We have seen in the exercise sheet 6 that the stress-energy tensor associated with the action of a point particle reads

$$T_1^{\mu\nu} = \frac{m u^\mu u^\nu}{u^0 \sqrt{-g}} \, \delta_{\mathrm{D}}[x^i - y^i(t)],$$

where $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau_g$. There is an important subtlety here: the quantity τ_g denotes proper time with respect to the metric $\boldsymbol{g}$, such that $\mathrm{d}\tau_g^2 = -g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+\Phi)^2 \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = (1+\Phi)^2 \mathrm{d}\tau^2$. Converting everything in terms of the Minkowski metric, we find

$$T_1^{\mu\nu} = \frac{m u^\mu u^\nu}{u^0 \sqrt{-g}} \, \delta_{\mathrm{D}}[x^i - y^i(t)] = \frac{m}{\gamma (1+\Phi)^5} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \, \delta_{\mathrm{D}}[x^i - y^i(t)],$$

where we used that $\sqrt{-g} = \sqrt{-\det (1+\Phi)^2 \eta_{\mu\nu}} = (1+\Phi)^4$, and assumed again that $1+\Phi \geq 0$. Its trace (with respect to the metric $\boldsymbol{g}$) thus reads

$$T_1 \equiv g_{\mu\nu} T_1^{\mu\nu} = -\frac{m}{\gamma (1+\Phi)^3} \, \delta_{\mathrm{D}}[x^i - y^i(t)]$$

Summing over all the particles, and comparing with the previous definition of ρ and P, we conclude that

$$T = -\frac{\rho - 3P}{(1+\Phi)^3}.$$

Q10. How can you rewrite Nordström's equation (6) in purely geometric terms? Compare with Einstein's equation.

Solution Q10

Using the relations between R and $\Box \Phi$, $\Box \Phi$ and $\rho - 3P$, as well as between $\rho - 3P$ and T, we can relate $R \sim T$ and find that Nordström's equation reads

$$R = 24 \pi G T.$$

It is, in some sense, a scalar version of Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

It is tempting to believe that Nordström's equation is just the trace of Einstein's equation. However, it is not the case, since the latter gives:

$$R = -8\pi G T,$$

which thus differs from the former by a factor -3.