Stress-Energy Tensor

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You have seen in the course that the energy and momentum of a matter field is described by its stress-energy tensor $\boldsymbol{T}$. This tensor typically encodes information about the energy density, the energy flux, and the momentum flux of matter. An important property of $\boldsymbol{T}$ is that it can be extracted directly from the action of matter. More precisely, consider a matter species—which we will represent here by a field $\boldsymbol{\psi}$—characterised by the action

$$S[\boldsymbol{\psi}, \boldsymbol{g}] = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}(\boldsymbol{\psi}, \boldsymbol{\nabla}\boldsymbol{\psi}, \boldsymbol{g}), \tag{1}$$

where g is the determinant of the metric, defined by

$$g = [\mu\nu\rho\sigma] g_{0\mu} g_{1\nu} g_{2\rho} g_{3\sigma} \tag{2}$$

with the permutation symbol $[\mu\nu\rho\sigma]$ such that $[0123] = 1$, while $\mathcal{L}$ denotes the covariant Lagrangian density of the matter field $\boldsymbol{\psi}$. The stress-energy tensor of $\boldsymbol{\psi}$ is the obtained by taking the variational derivative of the action S with respect to the metric:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad \text{that is} \qquad \delta S = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}. \tag{3}$$

In this exercise sheet, we propose to use this property to determine the stress-energy tensor of point particles, perfect fluids, and scalar fields.

1 From particles to fluids

In this exercise, we propose to determine the stress-energy tensor of a massive point particle, then a system of point particles, and see how we can recover the expression of the stress-energy tensor of a perfect fluid that you have seen in the course.

1.1 Point particle

Consider a point particle with mass m and neither spin not charge. The corresponding action is a function of the trajectory $\boldsymbol{y}(\lambda)$ of the particle through spacetime, as

$$S[\boldsymbol{y}, \boldsymbol{g}] = -m \int \mathrm{d}\tau = -m \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu} \dot{y}^\mu \dot{y}^\nu}, \tag{4}$$

where τ is the particle's proper time, while λ is an arbitrary parameter, and $\dot{y}^\mu \equiv \mathrm{d}y^\mu / \mathrm{d}\lambda$. If we want to extract the energy-stress tensor of such matter distribution, we have to face a first issue: the above integral concerns a single world-line, while eq. (1) is an integral over the whole spacetime, or at least over a 4-dimensional region of spacetime.

Q1. Show that one can actually write

$$S = \int \mathrm{d}^4x \, \mathcal{F}(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{y}}, \boldsymbol{g}), \tag{5}$$

where $\mathcal{F}$ is a distribution to be determined.

Solution Q1

An integral over spacetime can be introduced thanks to the Dirac distribution as

$$\begin{aligned} S &= -m \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu} \\ &= -m \int \mathrm{d}\lambda \int \mathrm{d}^4x \; \delta_\mathrm{D}[\boldsymbol{x} - \boldsymbol{y}(\lambda)] \sqrt{-g_{\mu\nu}[\boldsymbol{y}(\lambda)] \, \dot{y}^\mu(\lambda) \dot{y}^\nu(\lambda)} \\ &= \int \mathrm{d}^4x \; \int (-m) \mathrm{d}\lambda \; \delta_\mathrm{D}[\boldsymbol{x} - \boldsymbol{y}(\lambda)] \sqrt{-g_{\mu\nu}(\boldsymbol{x}) \dot{y}^\mu(\lambda) \dot{y}^\nu(\lambda)} \end{aligned}$$

where we can identify the distribution $\mathcal{F}$.

Q2. Take the variation of S with respect to a variation of the metric, and show that the resulting stress-energy tensor reads

$$T^{\mu\nu}(x^\mu) = \frac{p^\mu p^\nu}{p^0} \, \frac{\delta_\mathrm{D}[x^i - y^i(t)]}{\sqrt{-g}}, \tag{6}$$

where $t = x^0$, δ_D is the Dirac distribution, and $\boldsymbol{p}$ is the particle's four-momentum.

Solution Q2

The metric is only involved in the square root, so that the variation of the action reads

$$\begin{aligned} \delta S &= \int \mathrm{d}^4x \; \int (-m) \mathrm{d}\lambda \; \delta_\mathrm{D}[\boldsymbol{x} - \boldsymbol{y}(\lambda)] \delta \left(\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu} \right) \quad , \quad \delta \left(\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu} \right) = \frac{-\dot{y}^\mu \dot{y}^\nu \, \delta g_{\mu\nu}}{2\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu}} \\ &= \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} \; \underbrace{\left\{ \frac{m}{\sqrt{-g}} \int \mathrm{d}\lambda \; \delta_\mathrm{D}[\boldsymbol{x} - \boldsymbol{y}(\lambda)] \; \frac{\dot{y}^\mu \dot{y}^\nu}{\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu}} \right\}}_{T^{\mu\nu}(\boldsymbol{x})} \delta g_{\mu\nu} \end{aligned}$$

where we artificially introduced the metric determinant $\sqrt{-g}$ in order to identify the stress-energy tensor as in the definition (3). We can further simplify its expression by performing the integral over λ. For that purpose, we perform a change of variable $\lambda \mapsto y^0$, and use that $\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu} = \mathrm{d}\tau/\mathrm{d}\lambda$ to write

$$\begin{aligned} T^{\mu\nu} &= \frac{m}{\sqrt{-g}} \int \mathrm{d}\lambda \; \delta_\mathrm{D}[\boldsymbol{x} - \boldsymbol{y}(\lambda)] \; \frac{\dot{y}^\mu \dot{y}^\nu}{\sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu}} \qquad , \quad \left(\begin{cases} \lambda \mapsto y^0 \\ \sqrt{-g_{\mu\nu}\dot{y}^\mu \dot{y}^\nu} = \mathrm{d}\tau/\mathrm{d}\lambda \end{cases} \right) \\ &= \frac{m}{\sqrt{-g}} \int \mathrm{d}y^0 \; \delta_\mathrm{D}(x^0 - y^0) \, \delta_\mathrm{D}[x^i - y^i(y^0)] \, \frac{\mathrm{d}y^0}{\mathrm{d}\tau} \frac{\mathrm{d}y^\mu}{\mathrm{d}y^0} \frac{\mathrm{d}y^\nu}{\mathrm{d}y^0} \qquad , \quad \left(\tfrac{\mathrm{d}y^\mu}{\mathrm{d}\tau} = \tfrac{p^\mu}{m} \right) \\ &= \frac{\cancel{m}}{\sqrt{-g}} \int \mathrm{d}y^0 \; \delta_\mathrm{D}(x^0 - y^0) \, \delta_\mathrm{D}[x^i - y^i(x^0)] \, \frac{\cancel{p^0}}{\cancel{m}} \frac{p^\mu}{\cancel{p^0}} \frac{p^\nu}{p^0} \\ &= \frac{p^\mu p^\nu}{p^0} \, \frac{\delta_\mathrm{D}[x^i - y^i(t)]}{\sqrt{-g}}, \end{aligned}$$

where introduced $p^\mu = m\mathrm{d}y^\mu/\mathrm{d}\tau$, and substituted $t = x^0$. This gives the desired result.

One could be a bit lost in the multitude of parameters λ, τ, y^0, t in the above. The important is to keep track of the meaning of the whole calculation. In the initial delta function $\delta_\mathrm{D}(\boldsymbol{x} - \boldsymbol{y})$, $\boldsymbol{x}$ is a free position, representing the argument of the field $\boldsymbol{T}(\boldsymbol{x})$. In other words, we are trying to evaluate $\boldsymbol{T}$ at $\boldsymbol{x}$. The coordinates $\boldsymbol{y}$ have a different status: they represent a fixed world-line for the particle. The delta function thus encodes the fact that the stress-energy of the particle takes non-zero values only at the events intersecting its world-line, which definitely makes sense. The last step, which consisted in integrating over λ, must be seen as if we where following the particle's world-line until the associated event $\boldsymbol{y}(\lambda)$ reaches $y^0 = x^0$. This is the only point where there is a non-zero contribution to this integral. For computational simplicity, it is then natural to use y^0 to parametrise the world-line, which is the reason why we performed the change of variable $\lambda \mapsto y^0$.

Q3. Interpret physically each term of the expression of T^{00}.

Solution Q3

Using the result (6), we find that

$$T^{00}(t, x^i) = \frac{p^0\,\delta_{\mathrm{D}}[x^i - y^i(t)]}{\sqrt{-g}}, \tag{7}$$

which represents the energy density of the particle. We are not surprised to see the quantity $p^0 = \gamma m$, that is the energy of the particle, m being its rest energy, while γ contains the effect of its motion with respect to the coordinate system in which it is evaluated. The δ_{D} function tells us that the energy density is zero where the particle is not, and infinite where the particle is, which is due to its punctual character. Finally, the presence of the $\sqrt{-g}$ transforms the coordinate volume implicitly present in the Dirac distribution, since $[\delta_{\mathrm{D}}] = 1/[\mathrm{d}^3x]$, into a physical volume.

1.2 Fluid

Consider an ensemble of non-interacting point particles. Because they do not interact, the action of the system is just the sum of the actions of the individual particles, and hence so is its stress-energy tensor. Let $\mathcal{D}$ be a domain of $\mathcal{M}$ centred around an event E, and whose dimensions are small compared to the typical spacetime curvature radius. Within $\mathcal{D}$, all vectors can be considered to approximately belong to the same tangent space $\mathrm{T}_E\mathcal{M}$, in particular the four-momenta of the particles contained in $\mathcal{D}$, which can, thus, be summed.

Q4. Define the four-velocity $\bar{\boldsymbol{u}}$ of the barycentric frame of the particles in $\mathcal{D}$, as a function of their four-momenta.

Solution Q4

If there are $N_{\mathcal{D}}$ non-interactive particles, labelled by a in $\mathcal{D}$, the total four-momentum is given by:

$$\boldsymbol{P} = \sum_{a=1}^{N_{\mathcal{D}}} \boldsymbol{p}_a = \sum_{a=1}^{N_{\mathcal{D}}} m\boldsymbol{u}_a$$

We can then decompose each particle four-momentum relative to the four-velocity of the barycentric frame $\bar{\boldsymbol{u}}$, such that the spatial part of each four-momentum be orthogonal to $\bar{\boldsymbol{u}}$:

$$\sum_{a=1}^{N_{\mathcal{D}}} \boldsymbol{p}_a = \sum_{a=1}^{N_{\mathcal{D}}} E_a \bar{\boldsymbol{u}} + \sum_{a=1}^{N_{\mathcal{D}}} \boldsymbol{p}_a^{\perp}$$

where $E_a = \mathbf{p}_a \cdot \bar{\mathbf{u}}$ denotes the energy of particle a, and $\mathbf{p}_a^{\perp}$ represents its spatial momentum components orthogonal to $\bar{\mathbf{u}}$ in the barycentric frame. Given the barycentric condition, $\sum \mathbf{p}_a^{\perp} = \mathbf{0}$, this simplifies to:

$$\boldsymbol{P} = \left(\sum_{a=1}^{N_{\mathcal{D}}} E_a\right) \bar{\boldsymbol{u}} \implies \bar{\boldsymbol{u}} = \frac{\boldsymbol{P}}{\sum_{a=1}^{N_{\mathcal{D}}} E_a}$$

The proportionality factor here depends on energies, which implicitly depends on $\bar{\boldsymbol{u}}$. In order to have an expression for $\bar{\boldsymbol{u}}$ which only depends on the four-momenta, we can just use that the norm of $\bar{\boldsymbol{u}}$ must be -1, so that

$$-1 = \boldsymbol{g}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}) = \boldsymbol{g}\left(\frac{\boldsymbol{P}}{\sum\limits_{a=1}^{N_\mathcal{D}} E_a}, \frac{\boldsymbol{P}}{\sum\limits_{a=1}^{N_\mathcal{D}} E_a}\right) = \frac{\boldsymbol{g}(\boldsymbol{P}, \boldsymbol{P})}{\left(\sum\limits_{a=1}^{N_\mathcal{D}} E_a\right)^2} \Longrightarrow \left(\sum_{a=1}^{N_\mathcal{D}} E_a\right)^2 = -\boldsymbol{g}(\boldsymbol{P}, \boldsymbol{P})$$

so at the end we can write

$$\bar{\boldsymbol{u}} = \frac{\boldsymbol{P}}{\sqrt{-\boldsymbol{g}(\boldsymbol{P}, \boldsymbol{P})}} = \left(\sum_{a=1}^{N_\mathcal{D}} \boldsymbol{p}_a\right)\left[-\sum_{a,b=1}^{N_\mathcal{D}} \boldsymbol{g}(\boldsymbol{p}_a, \boldsymbol{p}_b)\right]^{-1/2}$$

Q5. Suppose that you work in a coordinate system adapted to this frame (e.g. the Fermi normal coordinate system associated with the associated world-line). Show that if $\boldsymbol{T}$ is smoothed over $\mathcal{D}$, then

$$T^{00} = n\,\langle \gamma m \rangle \tag{8}$$
$$T^{0i} = 0 \tag{9}$$
$$T^{ij} = n\langle \gamma m v^i v^j \rangle, \tag{10}$$

where $n = N_\mathcal{D}/V_\mathcal{D}$ is the number density of particles, and v^i the velocity of the particles, and γ the associated Lorentz factor.

Solution Q5

In the barycentric frame, if we work in a coordinate system such that the metric can be considered flat over $\mathcal{D}$, then $\sqrt{-g} = 1$, and the stress-energy tensor of the system reads

$$T^{\mu\nu} = \sum_{a=1}^{N_\mathcal{D}} \frac{p_a^\mu p_a^\nu}{p_a^0}\,\delta_\mathrm{D}[x^i - y_a^i(t)].$$

The four-momentum p_a^μ is nothing but $\gamma_a m_a(1, v_a^i)$, where v_a^i is the velocity of particle a as measured in the barycentric frame, and γ_a the corresponding Lorentz factor. The 00 component of $\boldsymbol{T}$ then reads

$$T^{00} = \sum_{a=1}^{N_\mathcal{D}} \gamma_a m_a\,\delta_\mathrm{D}[x^i - y_a^i(t)],$$

which, once smoothed over $\mathcal{D}$ (averaging the discrete quantities over the spatial domain $\mathcal{D}$) simply becomes

$$\begin{aligned}\langle T^{00}\rangle_\mathcal{D} &= \frac{1}{V_\mathcal{D}} \sum_{a=1}^{N_\mathcal{D}} \gamma_a m_a \int_\mathcal{D} \delta_D[x^i - y_a^i(t)]\,dV \quad , \quad \left(\int_\mathcal{D} \delta_D[x^i - y_a^i(t)]\,dV = \begin{cases} 1 & \text{if } y_a^i(t) \in \mathcal{D} \\ 0 & \text{otherwise}\end{cases}\right) \\ &= \frac{1}{V_\mathcal{D}} \sum_{a=1}^{N_\mathcal{D}} \gamma_a m_a \equiv n\,\langle \gamma m \rangle\,. \end{aligned}$$

where in the last step we used the fact that $\langle \gamma m \rangle = \frac{1}{N_\mathcal{D}} \sum\limits_{a=1}^{N_\mathcal{D}} \gamma_a m_a$ is the average of $\gamma_a m_a$ over all particles in the domain $\mathcal{D}$.

Similarly, we get $T^{0i} = n\,\langle \gamma m v^i \rangle = 0$ in the barycentric frame, and $T^{ij} = n\,\langle \gamma m v^i v^j \rangle$.

Q6. In which limit does one recover the stress-energy tensor of a perfect fluid?

Solution Q6

In the limit of a large number of particles with random velocities,

$$\left\langle \gamma m v^i v^j \right\rangle = \frac{1}{3} \left\langle \gamma m v^2 \right\rangle \delta^{ij}$$

since for $i \neq j$, $\langle \gamma m v^i v^j \rangle = \langle \gamma m v^i \rangle \langle v^j \rangle = 0$. We thus recover the expression of the stress-energy tensor of a perfect fluid with energy density ρ and kinetic pressure p, where

$$\rho \equiv n \langle \gamma m \rangle \qquad p = \frac{n}{3} \left\langle \gamma m v^2 \right\rangle$$

For non-relativistic particles ($v \ll 1$), $p \ll \rho$, while for ultra-relativistic particles ($v \to 1$), $p = \rho/3$, like in a gas of photons. The fact that γ is involved in the expression of ρ tells us about a very important difference between Einstein's gravity and Newton's gravity: in relativity, a hot gas is heavier than a cold gas!

2 Variation of the metric determinant

The variation of a generic action

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, \mathcal{L} \tag{11}$$

for a variation $\delta \boldsymbol{g}$ of the metric reads

$$\delta S = \int \mathrm{d}^4 x \, \delta \left(\sqrt{-g} \mathcal{L} \right) = \int \mathrm{d}^4 x \, \left(\mathcal{L} \, \delta \sqrt{-g} + \sqrt{-g} \, \delta \mathcal{L} \right), \tag{12}$$

it is thus generally useful to determine the variation $\delta\sqrt{-g}$ of the metric determinant, which we propose to do in this exercise. For that purpose, let us consider $\boldsymbol{g}$ just like any invertible matrix.

Q1. Justify that, for a variation $\delta \boldsymbol{g}$ of $\boldsymbol{g}$, the variation of its determinant g can be written

$$\delta g = g \left[\det(\mathbf{1} + \boldsymbol{g}^{-1} \delta \boldsymbol{g}) - 1 \right] \tag{13}$$

Solution Q1

We use that the determinant is a morphism with respect to matrix multiplication to write

$$\begin{aligned} \delta g &= \det(\boldsymbol{g} + \delta \boldsymbol{g}) - \det \boldsymbol{g} \\ &= \det[\boldsymbol{g}(\mathbf{1} + \boldsymbol{g}^{-1} \delta \boldsymbol{g})] - \det \boldsymbol{g} \\ &= \det \boldsymbol{g} \det(\mathbf{1} + \boldsymbol{g}^{-1} \delta \boldsymbol{g}) - \det \boldsymbol{g} \\ &= g \left[\det(\mathbf{1} + \boldsymbol{g}^{-1} \delta \boldsymbol{g}) - 1 \right] \end{aligned}$$

which yields the desired expression.

Q2. Show that, for any $n \times n$ matrix $\boldsymbol{M}$, $\det(\mathbf{1} + \varepsilon \boldsymbol{M}) = 1 + \varepsilon \, \mathrm{tr} \boldsymbol{M} + \mathcal{O}(\varepsilon^2)$.

Solution Q2

We start from the definition of the determinant of an $n \times n$ matrix $\boldsymbol{A}$ from the sum over permutations of indices,

$$\det \boldsymbol{A} = [i_1 i_2 \ldots i_n] \, A_{1 i_1} A_{2 i_2} \ldots A_{n i_n}.$$

If we apply the above to $\boldsymbol{A} = \boldsymbol{1} + \varepsilon \boldsymbol{M}$, i.e. $A_{ij} = \delta_{ij} + \varepsilon M_{ij}$, and expand the product at first order in ε, we find

$$\begin{aligned}\det(\boldsymbol{1} + \varepsilon \boldsymbol{M}) &= [i_1 i_2 \dots i_n]\,(\delta_{1i_1} + \varepsilon M_{1i_1})(\delta_{2i_2} + \varepsilon M_{2i_2}) \dots (\delta_{ni_n} + \varepsilon M_{ni_n}) \\ &= [i_1 i_2 \dots i_n] \delta_{1i_1} \delta_{2i_2} \dots \delta_{ni_n} \\ &\quad + \varepsilon\, [i_1 i_2 \dots i_n](M_{1i_1} \delta_{2i_2} \dots \delta_{ni_n} + \delta_{1i_1} M_{2i_2} \dots \delta_{ni_n} + \dots + \delta_{1i_1} \delta_{2i_2} \dots M_{ni_n}) \\ &\quad + \mathcal{O}(\varepsilon^2).\end{aligned}$$

In the $\mathcal{O}(1)$ term, the only non-zero permutation is the identity permutation, so that this term is just 1. The situation is similar for the $\mathcal{O}(\varepsilon)$ terms: each of them has $n-1$ delta functions, implying that the only non-zero permutation is also the identity. We are then left with

$$\begin{aligned}\det(\boldsymbol{1} + \varepsilon \boldsymbol{M}) &= 1 + \varepsilon(M_{11} + M_{22} + \dots M_{nn}) + \mathcal{O}(\varepsilon^2) \\ &= 1 + \varepsilon \,\mathrm{tr}\boldsymbol{M} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Q3. Conclude that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}. \tag{14}$$

Solution Q3

The inverse of $[g_{\mu\nu}]$ being $[g^{\mu\nu}]$, and using the previous results, we have

$$\delta g = g\left[\det(\boldsymbol{1} + \boldsymbol{g}^{-1}\delta\boldsymbol{g}) - 1\right] = g\left[1 + \mathrm{tr}(\boldsymbol{g}^{-1}\delta\boldsymbol{g}) - 1\right] = g\,\mathrm{tr}(\boldsymbol{g}^{-1}\delta\boldsymbol{g}) = g\, g^{\mu\nu}\delta g_{\mu\nu} \tag{15}$$

whence

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2\sqrt{-g}}(-g g^{\mu\nu}\delta g_{\mu\nu}) = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} \tag{16}$$

The alternative expression, involving the variation of the inverse metric, is obtained by taking the variation of $g^{\mu\nu} g_{\mu\nu} = 4$, that is $\delta g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0$.

3 Stress-energy tensor of a scalar field

The action of a scalar field ϕ minimally coupled to space-time geometry reads

$$S[\phi, \boldsymbol{g}] = \int \mathrm{d}^4x \sqrt{-g}\left[-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right], \tag{17}$$

where V is the self-interaction potential of the field.

Q1. Show that its stress-energy tensor reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial^\rho\phi\partial_\rho\phi)g_{\mu\nu} - V(\phi)g_{\mu\nu}. \tag{18}$$

Solution Q1

Taking the variation of the action yields:

$$\delta S = \int \mathrm{d}^4x\, \delta\left(\sqrt{-g}\mathcal{L}\right) = \int \mathrm{d}^4x\, \left(\mathcal{L}\,\delta\sqrt{-g} + \sqrt{-g}\,\delta\mathcal{L}\right), \tag{19}$$

We already know how the metric determinant transforms:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{20}$$

The Lagrangian contribution, besides, is very simple:

$$\delta\mathcal{L} = -\frac{1}{2}\delta\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\,\delta g^{\mu\nu} \tag{21}$$

putting these back into Eq. (19), we get

$$\delta S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left(\partial_\mu\phi\partial_\nu\phi + \mathcal{L}g_{\mu\nu}\right)\delta g^{\mu\nu}, \tag{22}$$

from which we directly identify the result. Beware of the sign! Since $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$, we have

$$T_{\mu\nu} = \ominus\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{23}$$

which takes a minus sign compared to the case of eq. (3) where we took the derivative with respect to $g_{\mu\nu}$.

Q2. Justify that there always exists, locally, a coordinate system in which the stress-energy tensor of ϕ takes the same form as a perfect fluid. Identify the corresponding energy density and pressure. *Hint:* remember that any metric is locally flat.

Solution Q2

Consider an iso-ϕ hyper-surface given by $\phi = \phi_0 = \text{cst}$. At any event E of this hyper-surface, we can define a normal vector $n_\mu \propto \partial_\mu\phi$. Consider a normal coordinate system (τ, X^i) corresponding to the frame with four-velocity $\boldsymbol{u} \propto \boldsymbol{n}$. In this coordinate system, in the vicinity of E, ϕ is homogeneous ($\partial_i\phi = 0$) and the metric is Minkowski, $g_{\alpha\beta} = \eta_{\alpha\beta}$. The stress-energy tensor of ϕ then reads

$$\begin{aligned}
T_{00} &= \frac{1}{2}(\partial_\tau\phi)^2 + V(\phi_0) \equiv \rho_\phi \\
T_{0i} &= 0 \\
T_{ij} &= \left[\frac{1}{2}(\partial_\tau\phi)^2 - V(\phi_0)\right]\delta_{ij} \equiv p_\phi\delta_{ij},
\end{aligned}$$

which has the form of a perfect fluid, with energy density ρ_ϕ and pressure p_ϕ.

Q3. Note that this pressure can be negative. Would such a thing be possible for a fluid made of massive particles? Do you know any context in which this property has been used?

Solution Q3

For $2V > (\partial_\tau\phi)^2$, we have $p_\phi < 0$. A scalar field is then analogous to a very bizarre fluid. Indeed, for a system of non-interacting massive particles, we have seen that the pressure reads $p = \langle\gamma m v^2\rangle/3 \geq 0$. Such a fluid with negative pressure is nonetheless quite useful in *cosmology*, in order to explain the two phases of accelerated expansion of our Universe: the primordial inflation on the one hand, and today's universe on the other hand. In the former context, the scalar field is called the inflaton; in the latter, it is called dark energy, or quintessence.