Geodesics and Parallel Transport

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

1 Geodesics

Let γ be a curve on $\mathcal{M}$. Let us introduce a parameter λ such that $x^\mu(\lambda)$ represents the coordinates of $\mathcal{M}$. The tangent vector $\boldsymbol{u}$ to γ is then such that $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. A *geodesic* is a self-parallel curve, i.e. whose tangent vector satisfies

$$\nabla_{\boldsymbol{u}}\boldsymbol{u} = f\boldsymbol{u}, \tag{1}$$

where f can be any function on $\mathcal{M}$.

Q1. Show that there exists a reparametrisation $\lambda \to s(\lambda)$ such that

$$\nabla_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{v}$ is the rescaled tangent vector, with $v^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}s$.

Solution Q1

If λ, s are two parameters for the same curve, then the associated tangent vectors are related by

$$v^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}\lambda}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \equiv \frac{1}{\dot{s}}u^\mu \ ,$$

where a dot denotes here a derivative with respect to λ. Using the algebraic properties of the connection, we then get

$$\nabla_{\boldsymbol{v}}\boldsymbol{v} = \nabla_{\boldsymbol{u}/\dot{s}}(\boldsymbol{u}/\dot{s}) = \frac{1}{\dot{s}}\left[\boldsymbol{u}\left(\frac{1}{\dot{s}}\right)\boldsymbol{u} + \frac{1}{\dot{s}}\nabla_{\boldsymbol{u}}\boldsymbol{u}\right] = \left(\frac{1}{\dot{s}^2}(f\dot{s} - \ddot{s})\right)\boldsymbol{v} \ ,$$

where in the last equality we used that

$$\boldsymbol{u}(\dot{s}) = u^\mu\partial_\mu(1/\dot{s}) = -\frac{1}{\dot{s}^2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial\dot{s}}{\partial x^\mu} = -\frac{1}{\dot{s}^2}\frac{\mathrm{d}\dot{s}}{\mathrm{d}\lambda} = -\frac{\ddot{s}}{\dot{s}^2} \ .$$

Therefore, if we chose s in such a way that $f\dot{s} - \ddot{s} = 0$, for example with

$$\dot{s}(\tau) = \exp\left\{\int_0^\lambda f[x^\mu(\lambda')]\,\mathrm{d}\lambda'\right\},$$

then $\nabla_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{0}$ as desired.

Q2. A parameter s which satisfies the above property is called an *affine parameter*. Why?

Solution Q2

If s is a parameter such that $\nabla_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{0}$, with $v^\mu = \mathrm{d}x^\mu/\mathrm{d}s$, then it is clear that for any other parameter of the form $s' = \alpha s + \beta$, where α and β are constant, we also have $\nabla_{\boldsymbol{v}'}\boldsymbol{v}' = \boldsymbol{0}$, with $(v')^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}s'$. In other words, any affine transformation of s leaves the special form of the geodesic equation unchanged, whence the name *affine parameter*.

Q3. Show that eq. (2) is equivalent to

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = 0. \tag{3}$$

Solution Q3

Let us introduce the components v^μ of the tangent vector $\boldsymbol{v}$ in the geodesic equation:

$$\mathbf{0} = \nabla_{\boldsymbol{v}}\boldsymbol{v} = \nabla_{v^\nu\boldsymbol{\partial}_\nu}\left(v^\rho\boldsymbol{\partial}_\rho\right) = (v^\nu\partial_\nu v^\mu)\boldsymbol{\partial}_\mu + v^\nu v^\rho\nabla_\nu\boldsymbol{\partial}_\rho = \left(\frac{\mathrm{d}v^\mu}{\mathrm{d}s} + v^\nu v^\rho\Gamma^\mu{}_{\nu\rho}\right)\boldsymbol{\partial}_\mu\ ,$$

from which we deduce

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}s} + v^\nu v^\rho\Gamma^\mu{}_{\nu\rho} = 0\ ,$$

and hence, substituting $v^\mu = \mathrm{d}x^\mu/\mathrm{d}s$,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = 0\ .$$

In pseudo Riemannian geometry, a geodesic is also a *stationary curve*, i.e. a curve whose total length is stationary with respect to small modifications of its path. Consider, for simplicity, a space-like curve; its total length between points A and B is the functional

$$S[x^\mu] = \int_A^B \mathrm{d}s = \int_A^B \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} = \int_A^B \mathrm{d}\lambda\,\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}. \tag{4}$$

Q4. Show that if S is stationary, that is if $\delta S/\delta x^\mu = 0$, then γ is indeed a self-parallel curve.

Solution Q4

The integrand of S is the Lagrangian of this optimisation problem,

$$L(x^\mu, \dot{x}^\mu) = \sqrt{g_{\mu\nu}(x^\rho)\dot{x}^\mu\dot{x}^\nu}\ ,$$

where we introduced the short-hand notation $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$. The functional derivative of S is then given by the usual Euler-Lagrange expression

$$\frac{\delta S}{\delta x^\mu} = \frac{\partial L}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\mu}\ .$$

Let us now calculate these derivatives. On the one hand

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2L}\,g_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho\ ;$$

on the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{L}\,g_{\mu\nu}\dot{x}^\nu\right) = -\frac{\dot{L}}{L^2}\,g_{\mu\nu}\dot{x}^\nu + \frac{1}{L}\,g_{\mu\nu,\rho}\dot{x}^\rho\dot{x}^\nu + \frac{1}{L}\,g_{\mu\nu}\ddot{x}^\nu\ .$$

and thus both results can be gathered as

$$\frac{\delta S}{\delta x^\mu} = -\frac{1}{L}\,g_{\mu\nu}\left[\ddot{x}^\nu + \frac{1}{2}\,g^{\nu\rho}\left(2g_{\rho\sigma,\tau} - g_{\sigma\tau,\rho}\right)\dot{x}^\sigma\dot{x}^\tau - \frac{\dot{L}}{L}\,\dot{x}^\nu\right].$$

Calling $u^\mu \equiv \dot{x}^\mu$, we recognise in the first two terms of the square bracket the covariant derivative of $\boldsymbol{u}$ along itself:

$$\begin{aligned}(\nabla_{\boldsymbol{u}}\boldsymbol{u})^\nu &= \frac{\mathrm{d}u^\nu}{\mathrm{d}\lambda} + \Gamma^\nu{}_{\sigma\tau}u^\sigma u^\tau \\ &= \frac{\mathrm{d}u^\nu}{\mathrm{d}\lambda} + \frac{1}{2}\, g^{\nu\rho}\left(g_{\rho\sigma,\tau} + g_{\rho\tau,\sigma} + g_{\sigma\tau,\rho}\right) u^\sigma u^\tau \\ &= \ddot{x}^\nu + \frac{1}{2}\, g^{\nu\rho}\left(2g_{\rho\sigma,\tau} + g_{\sigma\tau,\rho}\right)\dot{x}^\sigma \dot{x}^\tau \ .\end{aligned}$$

Therefore,

$$\frac{\delta S}{\delta x^\mu} = 0 \Longleftrightarrow \nabla_{\boldsymbol{u}}\boldsymbol{u} = \frac{\dot{L}}{L}\boldsymbol{u} \ ,$$

which is indeed the geodesic equation.

Q5. What is the geometric meaning of the affine parameter here?

Solution Q5

An affine parameter λ would be such that $\mathrm{d}L/\mathrm{d}\lambda = 0$, so that the right-hand side of the geodesic equation does disappear. A suitable choice is simply $\lambda = s$, so that $L = 1$. This corresponds to parametrising the curve with its own length.

Q6. How would the above change if we considered a time-like or a null curve instead?

Solution Q6

Along a time-like curve from A to B, an infinitesimal displacement $\mathrm{d}x^\mu$ is time-like, i.e. $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tau^2 < 0$. In that case, the total length of the curve must be replaced by the total duration of the trip AB, from the point of view of an observer following that curve

$$S = \int_A^B \mathrm{d}\tau = \int_A^B \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} = \int_A^B \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \ .$$

In other words, a time-like geodesics is a curve whose duration is stationary. The case of null curves is more subtle; since any displacement $\mathrm{d}x^\mu$ is then null, i.e. $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0$, a construction of the above form is practically impossible: it obviously does not make sense to try to make 0 stationary. However, viewing a null curve as a limit of a time-like or a space-like curve, it is still possible to formally write something like

$$S = \int_A^B \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \ ,$$

and get the correct geodesic equation, but this must really be considered a trick, with no geometrical meaning. In fact, there is an alternative and cleaner way to define null geodesics from a variational principle. Consider an event E, corresponding to the emission of a light signal, and an arbitrary observer receiving this signal; one can show that the signal follows a geodesic if and only if the time at which the observer receives the signal is stationary with respect to changes of the signal's trajectory. This result can be considered as a relativistic version of Fermat's principle. See e.g. § 1.2.5 of `[arXiv:1511.03702]` for further details.

2 Parallel transport

A vector field $\boldsymbol{v}$ is said to be parallely transported along a curve γ if its covariant derivative with respect to the tangent vector $\boldsymbol{u}$ of γ is zero,

$$\nabla_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{0}. \tag{5}$$

Q1. Show that a parallely transported vector has its norm conserved along γ.

Solution Q1

We know that the action of a generic tensor $\boldsymbol{g}$ (of type (0,2)) on 2 generic vectors, $\boldsymbol{u}$, $\boldsymbol{v}$ can be written as $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})$. This action results in a real number. The action of a metric on $(\boldsymbol{v},\boldsymbol{v})$ would then simply give the norm of the vector $\boldsymbol{v}$. Using the fact that the Levi-Civita connection of Riemannian geometry is metric preserving,

$$\boldsymbol{u}\left[\boldsymbol{g}(\boldsymbol{v},\boldsymbol{v})\right] = \boldsymbol{g}(\nabla_{\boldsymbol{u}}\boldsymbol{v},\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v},\nabla_{\boldsymbol{u}}\boldsymbol{v}) = 2\boldsymbol{g}(\nabla_{\boldsymbol{u}}\boldsymbol{v},\boldsymbol{v}) = 2\boldsymbol{g}(\boldsymbol{0},\boldsymbol{v}) = 0 \ ,$$

the derivative of the norm of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$ vanishes. Hence, the norm of $\boldsymbol{v}$ is conserved along γ. In the above equation, $\boldsymbol{u}\left[\boldsymbol{g}(\boldsymbol{v},\boldsymbol{v})\right]$ is the action of a vector on a real number (norm of $\boldsymbol{v}$). If one prefers to use components and indices, the proof goes as follows: let us denote λ an arbitrary parameter for the curve γ, then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\mu\nu}v^\mu v^\nu) = u^\rho \partial_\rho (g_{\mu\nu}v^\mu v^\nu) = u^\rho \nabla_\rho (g_{\mu\nu}v^\mu v^\nu) = u^\rho g_{\mu\nu;\rho}v^\mu v^\nu + 2u^\rho g_{\mu\nu}v^\mu {v^\nu}_{;\rho} = 0 + 0 \ .$$

Q2. Show that if γ is an affinely parametrised geodesic, then the scalar product between $\boldsymbol{u}$ and $\boldsymbol{v}$ is conserved as well.

Solution Q2

With a similar reasoning as in the previous question/solution,

$$\boldsymbol{u}\left[\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})\right] = \boldsymbol{g}(\nabla_{\boldsymbol{u}}\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{u},\nabla_{\boldsymbol{u}}\boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{0},\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{u},\boldsymbol{0}) = 0 \ ,$$

or, with components and indices,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\mu\nu}u^\mu v^\nu) = u^\rho (g_{\mu\nu}u^\mu v^\nu)_{;\rho} = u^\rho g_{\mu\nu;\rho}v^\mu v^\nu + u^\rho g_{\mu\nu}{u^\mu}_{;\rho}v^\nu + u^\rho g_{\mu\nu}u^\mu {v^\nu}_{;\rho} = 0 + 0 + 0 \ .$$

Consider the situation depicted on fig. 1. The vector $\boldsymbol{v}$, tangent to the sphere at its North pole, is parallely transported along the sphere[1] along the closed curve γ, made of portions of great circles.

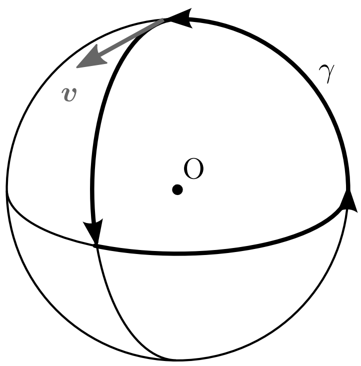

Figure 1 A vector $\boldsymbol{v}$ is parallely transported along the sphere following the closed curve γ.

[1] while keeping tangent to the sphere, i.e. the surface of the sphere is considered the manifold in which we define geometry and vector fields

Q3. Can you determine, without any calculation, what is going to be the state of $\boldsymbol{v}$ after being parallely transported along γ? Well, if you are not sure... calculate!

Solution Q3

The path γ is made of three segments which are all three portions of geodesics (great circles). Hence, from the two previous questions, we know that the length of the vector and its angle with respect to the tangent vector is preserved during transport. Along the first segment, from the north pole to the equator, $\boldsymbol{v}$ keeps aligned with the tangent vector $\boldsymbol{\partial}_\theta$, hence it ends up pointing to the south pole. Along the second portion, along the equator, $\boldsymbol{v}$ remains perpendicular to the equator, and ends up still pointing to the south pole. Finally, in the third segment, $\boldsymbol{v}$ is kept aligned with $\boldsymbol{\partial}_\theta$ again, and finishes turned 90° with respect to its initial state. The fact that $\boldsymbol{v}$ ends up turned after being parallely transported along a closed curve is the proof that the sphere is a curved manifold. Explicit calculations can be found, e.g., here.

3 Tensor: Some practise (more!)

This is a computational exercise. The goal is to practise the technical computations (Curvature, Parallel transport, etc.). You can skip it if you feel comfortable. **The three parts of the problem are independent.**

Part I

We consider the Cartesian coordinates $x^\mu = (t, x, y, z)$ and the metric

$$\boldsymbol{g} = -\mathbf{d}t^2 + f(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)\,, \quad f(t) > 0\,, \tag{6}$$

and the 4-vector

$$\boldsymbol{u} = \boldsymbol{\partial}_t + v(t)\boldsymbol{\partial}_x\,. \tag{7}$$

Q1. Compute the Christoffel symbols, the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$.

Hint: Compute the inverse metric $g^{\mu\nu}$ and use the symetries of the problem to reduce the number of computations you need to do.

Solution Q1

As the metric is diagonal and there is only a time dependance, computing these various quantities can be done in a straightforward was, and one obtains (Latin indices represents spatial indices)

$$\Gamma^t{}_{tt} = 0\,, \quad \Gamma^t{}_{ij} = \frac{f'}{2}\delta_{ij}\,, \tag{8}$$

$$\Gamma^i{}_{tt} = \Gamma^i{}_{jk} = 0\,, \quad \Gamma^i{}_{tj} = \Gamma^i{}_{jt} = \delta^i_j \frac{f'}{2f}\,, \tag{9}$$

$$R^t{}_{itj} = \left(\frac{f''}{2} - \frac{(f')^2}{4f}\right)\delta_{ij}\,, \quad R^i{}_{tjt} = \left(-\frac{f''}{2f} + \frac{(f')^2}{4f^2}\right)\delta_{ij}\,, \quad R^i{}_{jkl} = 0,\,, \tag{10}$$

$$R_{tt} = 3\frac{(f')^2 - 2ff''}{4f^2}\,, \quad R_{ij} = \frac{1}{4}\left(\frac{(f')^2}{f} + 2f''\right)\delta_{ij}\,, \tag{11}$$

$$R = \frac{3f''}{f}\,, \tag{12}$$

$$G_{tt} = \frac{3}{4}\frac{(f')^2}{f^2}\,, \quad G_{ij} = \left(\frac{(f')^2}{4f} - f''\right)\delta_{ij}\,, \tag{13}$$

$$\tag{14}$$

Q2. Compute $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$. Is this vector parallel transported?

Solution Q2

Using the index notation

$$u^\mu = (1, v(t), 0, 0) \tag{15}$$

and the definition

$$(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u})^\mu = u^\nu(\boldsymbol{\nabla}_\nu \boldsymbol{u})^\mu = u^\nu(\partial_\nu u^\mu + \Gamma^\mu{}_{\nu\alpha}u^\alpha) \tag{16}$$

one gets

$$(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u})^\mu = \left(\frac{1}{2}v^2 f', \frac{vf'}{f} + v', 0, 0\right) . \tag{17}$$

The vector is in general not parallel transported, except if $v = 0$, which would represent a particle at rest. Also, in the particular case

$$v(t) = \frac{1}{\sqrt{\alpha f^2 + f}} , \tag{18}$$

one has

$$\left(\frac{2f + 2\alpha f^2}{f'}\right)(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}) = \boldsymbol{u} , \tag{19}$$

which is the general definition of parallel transport.

Part II

We consider the unit sphere with coordinates $x^\mu = (\theta, \phi)$ and metric

$$\mathbf{d}s^2 = \mathbf{d}\theta^2 + \sin^2(\theta) + \mathbf{d}\phi^2 \tag{20}$$

We consider a particle whose trajectory is given by

$$x^\mu(\lambda) = (\theta(\lambda), \phi(\lambda)) = (\theta_0, \omega\lambda) , \quad \theta_0 \in (0, \pi) . \tag{21}$$

Q3. Compute $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$, where

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} .$$

Solution Q3

Using the same method as before, the result is

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = -\omega^2 \cos\theta \sin\theta \boldsymbol{\partial}_\theta . \tag{22}$$

Q4. For which value(s) of θ_0 is $\boldsymbol{u}$ parallel transported along itself? Is it surprising?

Solution Q4

The vector is parallel transported only if $\theta = \pi/2$ (excluding the poles). The particle travels on a horizontal cirlce (θ constant) and it is intuitive that only the largest circle is indeed a geodesic, which corresponds to the Equator $\theta = \pi/2$.

Roughly speaking, the quantity $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$ corresponds to the acceleration on a particle moving on a curved space. From a Newtonian perspective, this corresponds to a force. In our case (assuming $\theta < \pi/2$, so that the particle is on the North hemisphere), the particle feels a force in the $-\boldsymbol{\partial}_\theta$ direction. This force is necessary to keep the particle on its track, otherwise it would "fall" toward the Equator because of the centrifugal force.

Part III

We consider a 4-dimensional Minkowski space, whose metric is

$$\mathbf{d}s^2 = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2 \,, \tag{23}$$

and a 3-dimensional hypersurfac S implicitely given by

$$-t^2 + x^2 + y^2 + z^2 = -R^2 \,. \tag{24}$$

A possible parametrisation is then

$$t = R\cosh\chi \,, \tag{25}$$

$$x = R\sinh\chi\sin\theta\cos\phi \,, \tag{26}$$

$$y = R\sinh\chi\sin\theta\sin\phi \,, \tag{27}$$

$$z = R\sinh\chi\cos\theta \,, \tag{28}$$

with $\chi \in \mathbb{R}_+$, $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$.

Q5. Compute the induced metric on the hypersurface S, i.e. compute

$$\mathbf{d}s^2 = g_{ab}\mathbf{d}x^a\mathbf{d}x^b \,, \tag{29}$$

with $x^a = (\chi, \theta, \phi)$.

Solution Q5

$$x = 4 \tag{30}$$

The line element is given by

$$\mathbf{d}s^2 = g_{\mu\nu}\mathbf{d}x^\mu\mathbf{d}x^\nu = g_{\mu\nu}J^\mu{}_a J^\nu{}_b\mathbf{d}x^a\mathbf{d}x^b = J^{\mathrm{T}}{}_a{}^\mu g_{\mu\nu}J^\nu{}_b\mathbf{d}x^a\mathbf{d}x^b = (\boldsymbol{J}^{\mathrm{T}}\boldsymbol{g}\boldsymbol{J})_{ab}\mathbf{d}x^a\mathbf{d}x^b \tag{31}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$ (the method is however general and works for any metric). Hence, the induced metric can be identified as

$$g_{ab} = (\boldsymbol{J}^{\mathrm{T}}\boldsymbol{g}\boldsymbol{J})_{ab} \tag{32}$$

and can be obtained using matrix multiplication. This yields

$$\mathbf{d}s^2 = R^2(\mathbf{d}\chi^2 + \sinh^2\chi\mathbf{d}\theta^2 + \sinh^2\chi\sin^2\theta\mathbf{d}\phi^2) \,. \tag{33}$$

Q6. Can you give an interpretation for the metric you obtained?

Solution Q6

This metric looks like the spherical metric, where χ would be the radial coordinate, and (θ, ϕ) the angular coordinates. However, the $\sinh^2\chi$ is unusual. This represents a spherical space where the radius of the circle χ constant are *larger* than expected. This space is called *hyperbolic*. Such a space has a positive curvature (wheras the sphere has negative curvature). It is hard to give a 3D illustration, but it is sometimes compared to a saddle (in 2D). This space is relevant in Cosmology, where one can consider that the space itself (not the spacetime!) has a non-vanishing curvature.