Curvature

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1 Symmetries of the Riemann tensor

The Riemann tensor is the curvature tensor of the Levi-Civita connection. It is defined as

$$\boldsymbol{R} : \begin{array}{c} \Gamma(\mathcal{M})^3 \to \Gamma(\mathcal{M}) \\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} \end{array} \tag{1}$$

with $\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}$. Its components over the coordinate basis $\{\boldsymbol{\partial}_\mu\}$ are

$$\boldsymbol{R}(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu)\boldsymbol{\partial}_\sigma = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\boldsymbol{\partial}_\sigma = R^\rho{}_{\sigma\mu\nu}\boldsymbol{\partial}_\rho. \tag{2}$$

Q1. Show that $R^\sigma{}_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\rho\nu} - \partial_\nu\Gamma^\sigma{}_{\rho\mu} + \Gamma^\sigma{}_{\tau\mu}\Gamma^\tau{}_{\rho\nu} - \Gamma^\sigma{}_{\tau\nu}\Gamma^\tau{}_{\rho\mu}$.

Solution Q1

Let us apply the definition of the covariant derivative acting on vectors. On the one hand,

$$\nabla_\mu\nabla_\nu\boldsymbol{\partial}_\sigma = \partial_\mu(\nabla_\nu\boldsymbol{\partial}_\sigma) - \Gamma^\alpha{}_{\mu\nu}(\nabla_\alpha\boldsymbol{\partial}_\sigma) - \Gamma^\alpha{}_{\mu\sigma}(\nabla_\nu\boldsymbol{\partial}_\alpha) = \left(\partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\sigma} - \Gamma^\alpha{}_{\mu\sigma}\Gamma^\lambda{}_{\nu\alpha}\right)\boldsymbol{\partial}_\lambda$$

where in the last equality we used the definition of Christoffel coefficients $\nabla_\nu\boldsymbol{\partial}_\sigma = \Gamma^\lambda{}_{\nu\sigma}\boldsymbol{\partial}_\lambda$ and we factorise $\boldsymbol{\partial}_\lambda$ out. By simply exchanging μ and ν, we then get the second term $\nabla_\nu\nabla_\mu\boldsymbol{\partial}_\sigma$

$$\begin{aligned}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\boldsymbol{\partial}_\sigma &= \left(\partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\sigma} - \Gamma^\alpha{}_{\mu\sigma}\Gamma^\lambda{}_{\nu\alpha} - \partial_\nu\Gamma^\lambda{}_{\mu\sigma} + \Gamma^\alpha{}_{\nu\mu}\Gamma^\lambda{}_{\alpha\sigma} + \Gamma^\alpha{}_{\nu\sigma}\Gamma^\lambda{}_{\mu\alpha}\right)\boldsymbol{\partial}_\lambda \\ &= \left(\partial_\mu\Gamma^\lambda{}_{\nu\sigma} - \partial_\nu\Gamma^\lambda{}_{\mu\sigma} - \Gamma^\alpha{}_{\mu\sigma}\Gamma^\lambda{}_{\nu\alpha} + \Gamma^\alpha{}_{\nu\sigma}\Gamma^\lambda{}_{\mu\alpha}\right)\boldsymbol{\partial}_\lambda,\end{aligned}$$

where in the last equality we used the symmetry of the Christoffel coefficients. By identification with the definition of the components of $\boldsymbol{R}$, we obtain the desired result.

The components of the Riemann tensor enjoy a number of symmetry, namely

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \qquad \text{(definition)}, \tag{3}$$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \qquad (\nabla \text{ is metric preserving}), \tag{4}$$

$$R_{\mu[\nu\rho\sigma]} = 0 \qquad (\nabla \text{ is torsion free}). \tag{5}$$

Q2. Show that, given eqs. (3), (4),

$$(5) \iff \begin{cases} R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \\ R_{[\mu\nu\rho\sigma]} = 0 \end{cases} \tag{6}$$

Hint for ($\Leftarrow$): check that if $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$, then $R_{\mu[\nu\rho\sigma]} = -R_{\nu[\mu\rho\sigma]} = -R_{\rho[\nu\mu\sigma]} = -R_{\sigma[\nu\rho\mu]}$.

Solution Q2

As a preliminary step, it is useful to notice that, given the antisymmetry properties (3), (4),

$$R_{\mu[\nu\rho\sigma]} = \frac{1}{3!}\left(R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} - R_{\mu\nu\sigma\rho} - R_{\mu\rho\nu\sigma} - R_{\mu\sigma\rho\nu}\right) = \frac{1}{3}\left(R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho}\right).$$

In other words, antisymmetrising the last three indices here corresponds to summing their circular permutations only (and dividing the result by 3).

Given this property, let us start by showing that (5) $\Rightarrow R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$. As noticed above, $R_{\mu[\nu\rho\sigma]} = 0$ means that the sum of the circular permutations of the last three indices, while keeping the first index fixed, is zero. Applying this to the four possibilities for the first index (μ, ν, ρ, or σ), we thus have

$$\begin{aligned}
0 &= R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} \\
0 &= R_{\nu\mu\sigma\rho} + R_{\nu\sigma\rho\mu} + R_{\nu\rho\mu\sigma} \\
0 &= R_{\rho\mu\sigma\nu} + R_{\rho\sigma\nu\mu} + R_{\rho\nu\mu\sigma} \\
0 &= R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} + R_{\sigma\rho\mu\nu} \ .
\end{aligned}$$

Now, summing all four lines, and using the antisymmetries again, we find $0 = 2(R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu})$, which is the desired result.

Let us now show that (5) $\Rightarrow R_{[\mu\nu\rho\sigma]} = 0$. It is useful here to realize that the antisymmetrisation over four indices can be expressed as a sum of the antisymmetrisations over three indices. This can be shown explicitly by writing all the $4! = 24$ terms, and gathering them by groups starting with the same index:

$$\begin{aligned}
4!R_{[\mu\nu\rho\sigma]} = &\left(R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} - R_{\mu\nu\sigma\rho} - R_{\mu\rho\nu\sigma} - R_{\mu\sigma\rho\nu}\right) \\
&- \left(R_{\nu\mu\rho\sigma} + R_{\nu\rho\sigma\mu} + R_{\nu\sigma\mu\rho} - R_{\nu\mu\sigma\rho} - R_{\nu\rho\mu\sigma} - R_{\nu\sigma\rho\mu}\right) \\
&- \left(R_{\rho\nu\mu\sigma} + R_{\rho\mu\sigma\nu} + R_{\rho\sigma\nu\mu} - R_{\rho\nu\sigma\mu} - R_{\rho\mu\nu\sigma} - R_{\rho\sigma\mu\nu}\right) \\
&- \left(R_{\sigma\nu\rho\mu} + R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} - R_{\sigma\nu\mu\rho} - R_{\sigma\rho\nu\mu} - R_{\sigma\mu\rho\nu}\right) \\
= &\ 3!\left(R_{\mu[\nu\rho\sigma]} - R_{\nu[\mu\rho\sigma]} - R_{\rho[\nu\mu\sigma]} - R_{\sigma[\nu\rho\mu]}\right).
\end{aligned}$$

More generally, once we have understood this mechanism for four indices, it is easy to deduce that for n indices one has

$$T_{[\mu_1\mu_2\ldots\mu_n]} = \frac{1}{n}\left(T_{\mu_1[\mu_2\ldots\mu_n]} - T_{\mu_2[\mu_1\ldots\mu_n]} - \ldots - T_{\mu_n[\mu_2\ldots\mu_1]}\right).$$

In the case we are interested in, we then clearly see that if $R_{\mu[\nu\rho\sigma]} = 0$ then $R_{[\mu\nu\rho\sigma]} = 0$ as well.

We now turn to the converse part ($\Leftarrow$). We first show the properties suggested in the hint. Using both the antisymmetry within the first and last pair of indices, and the symmetry in the exchange of those pairs, we find

$$3R_{\nu[\mu\rho\sigma]} = R_{\nu\mu\rho\sigma} + R_{\nu\rho\sigma\mu} + R_{\nu\sigma\mu\rho} = -R_{\mu\nu\rho\sigma} - R_{\mu\sigma\nu\rho} - R_{\mu\rho\sigma\nu} = -3R_{\mu[\nu\rho\sigma]}.$$

The same kind of relations can be shown for $R_{\rho[\nu\mu\sigma]}$ and $R_{\sigma[\nu\rho\mu]}$, so that

$$0 = R_{[\mu\nu\rho\sigma]} = \frac{1}{4}\left(R_{\mu[\nu\rho\sigma]} - R_{\nu[\mu\rho\sigma]} - R_{\rho[\nu\mu\sigma]} - R_{\sigma[\nu\rho\mu]}\right) = R_{\mu[\nu\rho\sigma]}$$

which ends the derivation of the desired equivalence.

Q3. Deduce from those symmetries that the number of independent and non-zero components of the Riemann tensor in dimension d is

$$N = \frac{d^2(d^2-1)}{12}. \tag{7}$$

Hint: There are several ways to prove this; one of them leads to

$$N = \frac{\frac{d(d-1)}{2}\left[\frac{d(d-1)}{2}+1\right]}{2} - \binom{d}{4}.$$

Solution Q3

In order to evaluate the number of independent components $R_{\mu\nu\rho\sigma}$ of $\boldsymbol{R}$, it is convenient to see them as a set of a priori d^4 numbers with the following *independent* constraints

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} \\ R_{[\mu\nu\rho\sigma]} &= 0 \ . \end{aligned}$$

The first three conditions can be efficiently accounted for as follows: we can write $R_{\mu\nu\rho\sigma} = R_{IJ}$, where I and J represents each pair of indices. In principle, there should be d^2 independent values for each of those pairs, but because of the antisymmetry properties this number falls to $D = d(d-1)/2$. Furthermore, since R_{IJ} is symmetric, if there are D independent values for I, J, its total number of independent components is $D(D+1)/2$.

To that number, we finally have to subtract the number of independent constraints set by the last symmetry $R_{[\mu\nu\rho\sigma]} = 0$. This corresponds to the number of possibilities of picking four different indices out of a set of d values, i.e. $\binom{d}{4}$. Therefore, the number of independent components of the Riemann tensor reads

$$\begin{aligned} N &= \frac{D(D+1)}{2} - \binom{d}{4} \\ &= \frac{\frac{d(d-1)}{2}\left(\frac{d(d-1)}{2}+1\right)}{2} - \frac{d(d-1)(d-2)(d-3)}{4!} \\ &= \frac{d(d-1)}{24}\left\{3\left[d(d-1)+2\right] - (d-2)(d-3)\right\} \\ &= \frac{d^2(d^2-1)}{12} \ . \end{aligned}$$

The Ricci tensor $\boldsymbol{Ric} : (\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{Ric}(\boldsymbol{u}, \boldsymbol{v}) = R_{\mu\nu} u^\mu v^\nu$ is a form of trace of the Riemann tensor, in the sense that its components read $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$.

Q4. How many independent components does the Ricci tensor have in dimension d? Conclude that the Ricci tensor contains all the information about Riemann curvature for $d \leq 3$.

Solution Q4

The Ricci tensor is symmetric, thanks to the symmetry of the components of Riemann tensor by exchange of the pairs of indices, and to the symmetry of the metric:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = g^{\rho\sigma} R_{\sigma\nu\rho\mu} = g^{\sigma\rho} R_{\rho\nu\sigma\mu} = g^{\rho\sigma} R_{\rho\nu\sigma\mu} = R_{\nu\mu} \ . \tag{8}$$

The constraint $R_{[\mu\nu\rho\sigma]} = 0$ does not bring further information, since it is a traceless quantity. Therefore the Ricci tensor has in principle $d(d+1)/2$ independent components. Now we note that

$$N - \frac{d(d+1)}{2} = \frac{d(d+1)}{2}\left[\frac{d(d-1)}{6} - 1\right] \tag{9}$$

is negative or zero for $d \leq 3$; in other words, there is enough room in the Ricci tensor to account for all the degrees of freedom of the Riemann tensor when $d \leq 3$.

2 An alternative definition for the covariant derivative

Suppose a Riemannian manifold, $\mathcal{M}$, is embedded into Euclidean space ($\mathbb{R}^n$) via the mapping $\vec{\Psi} : \mathbb{R}^d \supset U \to \mathbb{R}^n$ such that the tangent space at $\vec{\Psi}(P)$ is spanned by the vectors

$$\left\{ \left. \frac{\partial \vec{\Psi}}{\partial x^i} \right|_P : i \in \{1, ..., d\} \right\} \tag{10}$$

and the scalar product on $\mathbb{R}^n$ is compatible with the metric on $\mathcal{M}$:

$$g_{ij} = \left\langle \frac{\partial \vec{\Psi}}{\partial x^i}, \frac{\partial \vec{\Psi}}{\partial x^j} \right\rangle . \tag{11}$$

Note that 10 is simply the basis of the tangent vector space at point P. Here, d is the dimension of the manifold with $d < n$.

Q1. The (contravariant) derivative of a metric g_{ab} is given by

$$\frac{\partial g_{ab}}{\partial x^c} = \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^c \, \partial x^a}, \frac{\partial \vec{\Psi}}{\partial x^b} \right\rangle + \left\langle \frac{\partial \vec{\Psi}}{\partial x^a}, \frac{\partial^2 \vec{\Psi}}{\partial x^c \, \partial x^b} \right\rangle \tag{12}$$

Using 12, show that

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2 \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} \right\rangle . \tag{13}$$

Solution Q1

$$\begin{aligned} &\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \\ &= \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j}, \frac{\partial \vec{\Psi}}{\partial x^k} \right\rangle + \left\langle \frac{\partial \vec{\Psi}}{\partial x^j}, \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^k} \right\rangle \\ &+ \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^j \, \partial x^k}, \frac{\partial \vec{\Psi}}{\partial x^i} \right\rangle + \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial^2 \vec{\Psi}}{\partial x^j \, \partial x^i} \right\rangle \\ &- \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^k \, \partial x^i}, \frac{\partial \vec{\Psi}}{\partial x^j} \right\rangle - \left\langle \frac{\partial \vec{\Psi}}{\partial x^i}, \frac{\partial^2 \vec{\Psi}}{\partial x^k \, \partial x^j} \right\rangle \\ &= 2 \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} \right\rangle \quad , \end{aligned}$$

where in the last line $\langle a, b \rangle = \langle b, a \rangle$ was used. The third and sixth terms as well as the second and fifth terms cancel.

A vector field ($\vec{V}$) in the tangent vector space of the manifold $\mathcal{M}$ can be written as

$$\vec{V} = v^j \frac{\partial \vec{\Psi}}{\partial x^j} \tag{14}$$

where, as mentioned before, $\partial \vec{\Psi} / \partial x^j$ is the basis of the tangent vector space. One has

$$\frac{\partial \vec{V}}{\partial x^i} = \frac{\partial v^j}{\partial x^i} \frac{\partial \vec{\Psi}}{\partial x^j} + v^j \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} \tag{15}$$

Q2. Do both the terms in 15 belong to the tangent space?

Solution Q2

No. While the first term is proportional to the vector $\frac{\partial \vec{\Psi}}{\partial x^j}$ and thus belongs to the tangent space, the second does not. See also the following question.

The second term in 15 can be expressed as

$$\frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j} = \Gamma^k{}_{ij} \frac{\partial \vec{\Psi}}{\partial x^k} + \vec{n} \tag{16}$$

This is a linear combination of the tangent space base vectors (using the Christoffel symbols as linear factors) and a vector orthogonal to the tangent space. The covariant derivative $\nabla_{\mathbf{e}_i} \vec{V}$, also written as $\nabla_i \vec{V}$, is defined as the orthogonal projection of the usual derivative onto tangent space:

$$\nabla_{\mathbf{e}_i} \vec{V} := \frac{\partial \vec{V}}{\partial x^i} - \vec{n} = \left(\frac{\partial v^k}{\partial x^i} + v^j \Gamma^k{}_{ij} \right) \frac{\partial \vec{\Psi}}{\partial x^k}. \tag{17}$$

Since $\vec{n}$ is orthogonal to tangent space, one can solve the normal equations:

$$\left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j}, \frac{\partial \vec{\Psi}}{\partial x^l} \right\rangle = \Gamma^k{}_{ij} \left\langle \frac{\partial \vec{\Psi}}{\partial x^k}, \frac{\partial \vec{\Psi}}{\partial x^l} \right\rangle = \Gamma^k{}_{ij} \, g_{kl} \tag{18}$$

Q3. Using 13 and 18, write the expression for the Christoffel symbols.

Solution Q3

Multiplying 18 with g^{lm}, gives

$$\begin{aligned} \Gamma^k{}_{ij} \, g_{kl} \, g^{lm} &= \Gamma^m{}_{ij} = \\ &= g^{lm} \left\langle \frac{\partial^2 \vec{\Psi}}{\partial x^i \, \partial x^j}, \frac{\partial \vec{\Psi}}{\partial x^l} \right\rangle = \\ &= \frac{g^{lm}}{2} \Big(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \Big) \ , \end{aligned}$$

which is the usual expression for the Christoffel symbols.

Q4. Why do you think it is more convenient to define the covariant derivative the way you did in class?

Solution Q4

We don't need to make any assumption about the manifold being embedded in a Euclidean space.

3 Sphere and cylinder

Consider a sphere with radius r embedded in the three-dimensional Euclidean space. This exercise proposes to study the geometric properties of the surface of the sphere, as a curved two-dimensional manifold. We will describe the surface of the sphere using spherical coordinates θ, φ.

Q1. What is the distance between two points of the sphere whose coordinates are (θ, φ) and $(\theta + \mathrm{d}\theta, \varphi + \mathrm{d}\varphi)$? Deduce from this the expression of the metric of the sphere, as

$$\mathrm{d}s^2 = g_{ab} \mathrm{d}\theta^a \mathrm{d}\theta^b = g_{\theta\theta} \mathrm{d}\theta^2 + 2 g_{\theta\varphi} \mathrm{d}\theta \mathrm{d}\varphi + g_{\varphi\varphi} \mathrm{d}\varphi^2. \tag{19}$$

Solution Q1

The sphere is embedded in the three-dimensional Euclidean space, for which the line element is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

Parametrising the position on the sphere with the two angles (θ, φ), we have

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad \text{so} \quad \begin{cases} \mathrm{d}x = r\,(\cos\theta\cos\varphi\,\mathrm{d}\theta - \sin\theta\sin\varphi\,\mathrm{d}\varphi) \\ \mathrm{d}y = r\,(\cos\theta\cos\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}\varphi) \\ \mathrm{d}z = -r\sin\theta\,\mathrm{d}\theta \end{cases}$$

Substituting in the three-dimensional Euclidean line-element, we then find

$$\mathrm{d}s^2 = r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right),$$

from which we read $g_{\theta\theta} = r^2$, $g_{\theta\varphi} = 0$, $g_{\varphi\varphi} = r^2\sin^2\theta$.

Q2. Determine the Christoffel symbols associated with this metric.

Solution Q2

There are a priori 6 independent Christoffel symbols $\Gamma^a{}_{bc} = g^{ad}\Gamma_{dbc}$. These are calculated as

$$\begin{aligned}
\Gamma_{\theta\theta\theta} &= \frac{1}{2}\left(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}\right) = 0 \\
\Gamma_{\theta\theta\varphi} &= \frac{1}{2}\left(g_{\theta\theta,\varphi} + g_{\theta\varphi,\theta} - g_{\theta\varphi,\theta}\right) = 0 \\
\Gamma_{\theta\varphi\varphi} &= \frac{1}{2}\left(g_{\theta\varphi,\varphi} + g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta}\right) = -r^2\cos\theta\sin\theta \\
\Gamma_{\varphi\theta\theta} &= \frac{1}{2}\left(g_{\varphi\theta,\theta} + g_{\varphi\theta,\theta} - g_{\theta\theta,\varphi}\right) = 0 \\
\Gamma_{\varphi\theta\varphi} &= \frac{1}{2}\left(g_{\varphi\theta,\varphi} + g_{\varphi\varphi,\theta} - g_{\theta\varphi,\varphi}\right) = r^2\cos\theta\sin\theta \\
\Gamma_{\varphi\varphi\varphi} &= \frac{1}{2}\left(g_{\varphi\varphi,\varphi} + g_{\varphi\varphi,\varphi} - g_{\varphi\varphi,\varphi}\right) = 0\ .
\end{aligned}$$

The only non-zero Christoffel symbols are therefore

$$\begin{aligned}
\Gamma^\theta{}_{\varphi\varphi} &= -\cos\theta\sin\theta\ , \\
\Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}\ .
\end{aligned}$$

Q3. Identify the only non-zero component of the Riemann tensor, and calculate its value.

Solution Q3

In two dimensions, the Riemann tensor has only one non-vanishing independent component:

$$\frac{d^2(d^2-1)}{12} = \frac{4\times 3}{12} = 1\ .$$

For all the indices downstairs, this component is $R_{\theta\varphi\theta\varphi} = -R_{\theta\varphi\varphi\theta} = -R_{\varphi\theta\theta\varphi} = R_{\varphi\theta\varphi\theta}$. Using the definition of this component and our results for the Christoffel symbols, we find

$$\begin{aligned}
R^\theta{}_{\varphi\theta\varphi} &= \partial_\theta\Gamma^\theta{}_{\varphi\varphi} - \partial_\varphi\Gamma^\theta{}_{\varphi\theta} + \Gamma^\theta{}_{a\theta}\Gamma^a{}_{\varphi\varphi} - \Gamma^\theta{}_{a\varphi}\Gamma^a{}_{\varphi\theta} \\
&= \partial_\theta\Gamma^\theta{}_{\varphi\varphi} - \Gamma^\theta{}_{\varphi\varphi}\Gamma^\varphi{}_{\varphi\theta} \\
&= \sin^2\theta\ ,
\end{aligned}$$

and hence $R_{\theta\varphi\theta\varphi} = g_{\theta\theta}R^\theta{}_{\varphi\theta\varphi} = r^2\sin^2\theta$.

Q4. Calculate the Ricci scalar, and compare with the quantity that you would expect for the curvature of a sphere.

Solution Q4

The Ricci scalar reads by definition

$$R = g^{ac}g^{bd}R_{abcd} = g^{11}g^{11}R_{1111} + g^{11}g^{22}R_{1212} + g^{22}g^{11}R_{2121} + g^{22}g^{22}R_{2222} = 2g^{11}g^{22}R_{1212} = \frac{2}{r^2} \, .$$

This is essentially what we would expect from a sphere: a homogeneous curvature related to the inverse of r. It is worth stressing that R has the dimension of the inverse of a length squared, hence it is related to the squared inverse of the curvature radius, here r.

Consider a cylinder with radius r about the z-axis of the three-dimensional Euclidean space.

Q5. Show that the metric on the surface of the cylinder reads

$$\mathrm{d}s^2 = r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \tag{20}$$

Solution Q5

On a cylinder with axis z and radius r, the Cartesian components of a point are constrained to read

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad \text{so} \qquad \begin{cases} \mathrm{d}x = -r\sin\varphi\,\mathrm{d}\varphi \\ \mathrm{d}y = r\cos\varphi\,\mathrm{d}\varphi \\ \mathrm{d}z = \mathrm{d}z \end{cases}$$

as r must remained fixed. Replacing in the three-dimensional Euclidean line element, we find

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 \ ,$$

so that the components of the metric are $g_{\varphi\varphi} = r^2$, $g_{\varphi z} = 0$, $g_{zz} = 1$.

Q6. Determine the Riemann tensor for this surface. What do you think about this result?

Solution Q6

The metric being homogeneous on the cylinder, the Christoffel symbols vanish, and hence so do the components of the Riemann tensor. This means that the cylinder is *not* an intrinsically curved surface: it is a Euclidean two-dimensional surface. By the way, we could have noticed it simply from a coordinate transformation $(\varphi, z) \mapsto (X = R\varphi, Y = z)$, leading to $\mathrm{d}s^2 = \mathrm{d}X^2 + \mathrm{d}Y^2$. In fact, the cylinder can be shown to have what is called an extrinsic curvature with respect to the three-dimensional space in which it is embedded, but if you were living on a cylinder with no access to the third dimension, you could never measure any curvature.