Connection and Metric

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1 Connection and covariant derivative

A connection ∇ on a manifold $\mathcal{M}$ is a structure which allows one to transport vectors, forms, and tensors from event to event. Let $\boldsymbol{u}$ be a vector field over $\mathcal{M}$, then $\nabla_{\boldsymbol{u}}$ is called the *covariant derivative* along $\boldsymbol{u}$, and can act on functions, vectors, forms, and tensors.

The effect of $\nabla_{\boldsymbol{u}}$ on a function f is simply

$$\nabla_{\boldsymbol{u}} f \equiv \boldsymbol{u}(f) \tag{1}$$

that is, the derivative of f along $\boldsymbol{u}$. The action of $\nabla_{\boldsymbol{u}}$ on vectors satisfies the following properties: for any function f and any vectors fields $\boldsymbol{v}, \boldsymbol{w}$,

$$\nabla_{\boldsymbol{u}+f\boldsymbol{v}} \boldsymbol{w} = \nabla_{\boldsymbol{u}} \boldsymbol{w} + f \nabla_{\boldsymbol{v}} \boldsymbol{w}, \tag{2}$$

$$\nabla_{\boldsymbol{u}}(\boldsymbol{v}+\boldsymbol{w}) = \nabla_{\boldsymbol{u}} \boldsymbol{v} + \nabla_{\boldsymbol{u}} \boldsymbol{w}, \tag{3}$$

$$\nabla_{\boldsymbol{u}}(f\boldsymbol{v}) = \boldsymbol{u}(f)\boldsymbol{v} + f \nabla_{\boldsymbol{u}} \boldsymbol{v}. \tag{4}$$

If we restrict to the action on vectors, ∇ can thus be seen as a function which eats two vectors and returns a vector, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \nabla_{\boldsymbol{u}} \boldsymbol{v}$.

Q1. From this point of view, is ∇ a tensor? Why?

Solution Q1

$\nabla : (\boldsymbol{u}, \boldsymbol{v}) \mapsto \nabla_{\boldsymbol{u}} \boldsymbol{v}$ is *not* a tensor, because it is not linear with respect to its second entry, by virtue of eq. (4)—one cannot simply 'take out' a function f mutliplying $\boldsymbol{v}$.

Let $\{x^\mu\}$ be a coordinate system on $\mathcal{M}$. One can define a notion of components for ∇ with respect to the coordinate basis $\{\boldsymbol{\partial}_\mu\}$, as

$$\nabla_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu \equiv \Gamma^\rho{}_{\nu\mu} \boldsymbol{\partial}_\rho, \tag{5}$$

where the numbers $\Gamma^\rho{}_{\mu\nu}$ are called the *connection coefficients*. The covariant derivative with respect to the coordinate basis $\nabla_{\boldsymbol{\partial}_\mu}$ is usually denoted ∇_μ for short.

Q2. Show that the covariant derivative of a vector field reads

$$\nabla_\mu \boldsymbol{v} = \left(\partial_\mu v^\nu + \Gamma^\nu{}_{\rho\mu} v^\rho \right) \boldsymbol{\partial}_\nu. \tag{6}$$

The component $(\nabla_\mu \boldsymbol{v})^\nu$ is usually denoted $\nabla_\mu v^\nu$ for short.

Solution Q2

We expand $\boldsymbol{v}$ over the coordinate basis as $\boldsymbol{v} = v^\nu \boldsymbol{\partial}_\nu$, and apply eq. (4) to get

$$\nabla_\mu \boldsymbol{v} \equiv \nabla_{\boldsymbol{\partial}_\mu}(v^\nu \boldsymbol{\partial}_\nu) = (\partial_\mu v^\nu) \boldsymbol{\partial}_\nu + v^\nu \nabla_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu.$$

Then, we use the definition of the connection coefficients to rewrite the second term:

$$v^\nu \nabla_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = v^\nu \Gamma^\rho{}_{\nu\mu} \boldsymbol{\partial}_\rho = v^\rho \Gamma^\nu{}_{\rho\mu} \boldsymbol{\partial}_\nu,$$

where in the second equality we just inverted the names of the mute indices ν and ρ. This yields the desired result

$$\nabla_\mu \boldsymbol{v} = \left(\partial_\mu v^\nu + \Gamma^\nu{}_{\rho\mu} v^\rho\right) \boldsymbol{\partial}_\nu.$$

Q3. How do the coefficients $\Gamma^\rho{}_{\nu\mu}$ change under a coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$?

Solution Q3

Since ∇ is not a tensor, its components $\Gamma^\rho{}_{\nu\mu}$ should not change according to the usual relation under a coordinate transformation:

$$\Gamma^\gamma{}_{\beta\alpha} \neq \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \Gamma^\rho{}_{\nu\mu}$$

in general. So in order to determine the transformation law, we must start from the definition of the connection coefficients and write

$$\nabla_\alpha \boldsymbol{\partial}_\beta = \Gamma^\gamma{}_{\beta\alpha} \boldsymbol{\partial}_\gamma.$$

Then, we use the algebraic properties of ∇ to introduce the coordinates system $\{x^\mu\}$ step by step. First, using eq. (4), we have

$$\nabla_\alpha \boldsymbol{\partial}_\beta = \nabla_\alpha \left(\frac{\partial x^\nu}{\partial y^\beta} \boldsymbol{\partial}_\nu\right) = \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \boldsymbol{\partial}_\nu + \frac{\partial x^\nu}{\partial y^\beta} \nabla_\alpha \boldsymbol{\partial}_\nu.$$

Second, by linearity with respect to the first entry of the connection, we can write

$$\nabla_\alpha \boldsymbol{\partial}_\nu \equiv \nabla_{\boldsymbol{\partial}_\alpha} \boldsymbol{\partial}_\nu = \nabla_{\frac{\partial x^\mu}{\partial y^\alpha} \boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = \frac{\partial x^\mu}{\partial y^\alpha} \nabla_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = \frac{\partial x^\mu}{\partial y^\alpha} \Gamma^\rho{}_{\nu\mu} \boldsymbol{\partial}_\rho.$$

Combining the last three equations, modulo a slight reorganisation of indices, we get

$$\Gamma^\gamma{}_{\beta\alpha} \boldsymbol{\partial}_\gamma = \left(\frac{\partial^2 x^\rho}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \Gamma^\rho{}_{\nu\mu}\right) \boldsymbol{\partial}_\rho.$$

The last step consists in replacing $\boldsymbol{\partial}_\rho$ by $(\partial y^\gamma / \partial x^\rho) \boldsymbol{\partial}_\gamma$, which gives

$$\Gamma^\gamma{}_{\beta\alpha} = \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial y^\alpha \partial y^\beta} + \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \Gamma^\rho{}_{\nu\mu}.$$

The action of the covariant derivative $\nabla_{\boldsymbol{u}}$ can be extended to differential forms. For that purpose, we assume that if $\boldsymbol{\omega}$ is a one-form, then $\nabla_{\boldsymbol{u}} \boldsymbol{\omega}$ is also a one-form, with

$$\forall \boldsymbol{v} \in \Gamma(\mathcal{M}) \qquad \nabla_{\boldsymbol{u}} [\boldsymbol{\omega}(\boldsymbol{v})] = (\nabla_{\boldsymbol{u}} \boldsymbol{\omega})(\boldsymbol{v}) + \boldsymbol{\omega}(\nabla_{\boldsymbol{u}} \boldsymbol{v}). \tag{7}$$

Q4. Derive the expression for the components $\nabla_\mu \omega_\nu$, a short-hand notation for $(\nabla_\mu \boldsymbol{\omega})_\nu$. Deduce the expression of $\nabla_\mu \mathbf{d}x^\sigma$.

Solution Q4

Since $\nabla_\mu \boldsymbol{\omega}$ is a one-form, its components read by definition $(\nabla_\mu \boldsymbol{\omega})_\nu = (\nabla_\mu \boldsymbol{\omega})(\boldsymbol{\partial}_\nu)$. From eq. (7), we then get

$$\nabla_\mu \omega_\nu = \nabla_\mu [\boldsymbol{\omega}(\boldsymbol{\partial}_\nu)] - \boldsymbol{\omega}(\nabla_\mu \boldsymbol{\partial}_\nu).$$

On the one hand, since $\boldsymbol{\omega}(\boldsymbol{\partial}_\nu) = \omega_\nu$ is simply a function, ∇_μ acts on it as

$$\nabla_\mu \left[\boldsymbol{\omega}(\boldsymbol{\partial}_\nu)\right] = \partial_\mu \omega_\nu;$$

on the other hand

$$\boldsymbol{\omega}(\nabla_\mu \boldsymbol{\partial}_\nu) = \boldsymbol{\omega}(\Gamma^\rho{}_{\nu\mu}\boldsymbol{\partial}_\rho) = \Gamma^\rho{}_{\nu\mu}\boldsymbol{\omega}(\boldsymbol{\partial}_\rho) = \Gamma^\rho{}_{\nu\mu}\omega_\rho;$$

whence

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho{}_{\nu\mu}\omega_\rho.$$

This result gives also the answer to the second question, if we apply it to $\boldsymbol{\omega} = \mathbf{d}x^\sigma$. In this case $\omega_\nu = \delta^\sigma_\nu$, and hence

$$\nabla_\mu \mathbf{d}x^\sigma = \left(\partial_\mu \delta^\sigma_\nu - \Gamma^\rho{}_{\nu\mu}\delta^\sigma_\rho\right)\mathbf{d}x^\nu = -\Gamma^\sigma{}_{\nu\mu}\mathbf{d}x^\nu$$

The action of $\nabla_{\boldsymbol{u}}$ can even be further extended to tensors, assuming that for any tensors $\boldsymbol{X}, \boldsymbol{Y}$,

$$\nabla_{\boldsymbol{u}}(\boldsymbol{X} \otimes \boldsymbol{Y}) = (\nabla_{\boldsymbol{u}}\boldsymbol{X}) \otimes \boldsymbol{Y} + \boldsymbol{X} \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{Y}). \tag{8}$$

Q5. Show that for any rank-(m, n) tensor:

$$\begin{aligned}\nabla_\rho X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} &\equiv (\nabla_\rho \boldsymbol{X})^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} \qquad (9)\\ &= \partial_\rho X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} + \Gamma^{\mu_1}{}_{\sigma\rho}X^{\sigma\dots\mu_n}{}_{\nu_1\dots\nu_m} + \dots + \Gamma^{\mu_n}{}_{\sigma\rho}X^{\mu_1\dots\sigma}{}_{\nu_1\dots\nu_m}\\ &\quad - \Gamma^\sigma{}_{\nu_1\rho}X^{\mu_1\dots\mu_n}{}_{\sigma\dots\nu_m} - \dots - \Gamma^\sigma{}_{\nu_m\rho}X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\sigma}. \qquad (10)\end{aligned}$$

Solution Q5

Let $\boldsymbol{X} = X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}$ be an (m, n)-tensor. It is quite clear, from the various Leibnitz laws satisfied by the covariant derivative, that

$$\begin{aligned}\nabla_\rho \boldsymbol{X} = (\partial_\rho X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m})\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}&\\ + X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}\nabla_\rho(\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m})&.\end{aligned}$$

We now have to figure out what is the second term. It is actually quite straightforward given eq. (8). The covariant derivative successively hits each vector and form of the tensor product, returning $m + n$ terms as

$$\begin{aligned}\nabla_\rho(\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}) &= \nabla_\rho\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}\\ &+ \dots\\ &+ \boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \nabla_\rho\boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}\\ &+ \boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \nabla_\rho\mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}\\ &+ \dots\\ &+ \boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \nabla_\rho\mathbf{d}x^{\nu_m}.\end{aligned}$$

From the previous questions, we know exactly how ∇_ρ acts on $\boldsymbol{\partial}_\mu$ and $\mathbf{d}x^\mu$; in particular

$$\begin{aligned}(\dots \otimes \nabla_\rho\boldsymbol{\partial}_{\mu_i} \otimes \dots) &= \Gamma^\sigma{}_{\mu_i\rho}(\dots \otimes \boldsymbol{\partial}_\sigma \otimes \dots)\\ (\dots \otimes \nabla_\rho\mathbf{d}x^{\nu_j} \otimes \dots) &= -\Gamma^{\nu_j}{}_{\sigma\rho}(\dots \otimes \mathbf{d}x^\sigma \otimes \dots).\end{aligned}$$

Because all the $m + n$ terms of the above form are multiplied by the components $X^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}$, we can exchange in each of them the mute indices μ_i or ν_j with σ, so that the basis tensor $\boldsymbol{\partial}_{\mu_1} \otimes \dots \otimes \boldsymbol{\partial}_{\mu_n} \otimes \mathbf{d}x^{\nu_1} \otimes \dots \otimes \mathbf{d}x^{\nu_m}$ factorises out to give the desired result.

2 Metric and inverse metric

In Riemannian (or pseudo-Riemannian) geometry, the manifold $\mathcal{M}$ is not only equipped with a connection, but also with a metric tensor $\boldsymbol{g} = g_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$, which allows one to define the *scalar product* between two vectors $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu}u^\mu v^\nu$, and hence angles and distances.

Q1. How does the metric coefficients change under a coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$?

Solution Q1

Because by definition the metric is a tensor, the usual transformation applies:

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}.$$

Q2. Consider the metric given by the line element

$$\begin{aligned} \mathrm{d}s^2 &\equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu & (11)\\ &= -(1-\Omega^2 r^2\sin^2\theta)\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + 2\Omega r^2 \sin^2\theta \mathrm{d}t\mathrm{d}\varphi + r^2\sin^2\theta\mathrm{d}\varphi^2, & (12)\end{aligned}$$

where Ω is a constant. Show that this is actually the Minkowski metric.

Solution Q2

Defining the angle $\psi \equiv \varphi + \Omega t$, we find that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

The usual form of the Minkowski metric is then recovered by changing the spatial coordinates from spherical (r, θ, φ) to Cartesian $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$, which yields $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Thus, the expression proposed in eq. (12) is the Minkowski metric written in a frame uniformly rotating about the z-axis.

As any scalar product, the metric provides a notion of duality between vectors and one-forms (different from the duality between $\boldsymbol{\partial}_\mu$ and $\mathbf{d}x^\mu$). Indeed, given a vector field $\boldsymbol{u}$ one can define the form $\boldsymbol{\eta_u} = \boldsymbol{g}(\boldsymbol{u}, \cdot)$, i.e., a map which eats a vector and returns its scalar product with $\boldsymbol{u}$, $\boldsymbol{\eta_u} : \boldsymbol{v} \mapsto \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$. Conversely, to a form $\boldsymbol{\omega}$ we can associate a vector $\boldsymbol{e}^{\boldsymbol{\omega}}$ such that $\boldsymbol{\omega} = \boldsymbol{g}(\boldsymbol{e}^{\boldsymbol{\omega}}, \cdot)$.

Q3. If we use the same notation for the components of $\boldsymbol{u}$ and those of its dual form $\boldsymbol{\eta_u}$, except for the position of the index, i.e., if we write $\boldsymbol{\eta_u} = u_\mu \mathbf{d}x^\mu$, show that the metric can be seen as the *index lowerer*.

Solution Q3

By definition of the one-form $\boldsymbol{\eta_u}$, we have

$$u_\mu = \boldsymbol{\eta_u}(\boldsymbol{\partial}_\mu) = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{\partial}_\mu) = \boldsymbol{g}(u^\nu\boldsymbol{\partial}_\nu, \boldsymbol{\partial}_\mu) = u^\nu \boldsymbol{g}(\boldsymbol{\partial}_\nu, \boldsymbol{\partial}_\mu) = u^\nu g_{\nu\mu},$$

whence the expression according to which the metric lowers indices.

Q4. Similarly, if we write $\boldsymbol{e}^{\boldsymbol{\omega}} = \omega^\mu \boldsymbol{\partial}_\mu$, show that the *index raiser* is the inverse of the metric.

Solution Q4

Let us take the scalar product of $\boldsymbol{e}^{\boldsymbol{\omega}}$ with the basis vector $\boldsymbol{\partial}_\mu$; on the one hand

$$\boldsymbol{g}(\boldsymbol{e}^{\boldsymbol{\omega}}, \boldsymbol{\partial}_\mu) = g_{\nu\mu}(\boldsymbol{e}^{\boldsymbol{\omega}})^\nu = g_{\nu\mu}\omega^\nu \;;$$

on the other hand, by definition of $\boldsymbol{e}^{\boldsymbol{\omega}}$,

$$\boldsymbol{g}(\boldsymbol{e}^{\boldsymbol{\omega}}, \boldsymbol{\partial}_\mu) = \boldsymbol{\omega}(\boldsymbol{\partial}_\mu) = \omega_\mu \ .$$

whence $\omega_\mu = g_{\mu\nu}\omega^\mu$. If we multiply both sides with the inverse $g^{\mu\nu}$ of the metric, in the sense of matrix products, that is such that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$, then we finally have

$$\omega^\mu = g^{\mu\nu}\omega_\nu \ ,$$

showing that the inverse metric plays the role of the index raiser.

Q5. Simplify the expression $g_{\mu\lambda}g^{\nu\sigma}g^{\lambda\tau}R^\mu{}_{\nu\rho\sigma}A_\tau$, where $R^\mu{}_{\nu\rho\sigma}$, A_τ are the components of a tensor and a form, respectively.

Solution Q5

Applying the rules of raising and lowering indices, as well as the definition of the inverse metric,

$$g_{\mu\lambda}g^{\nu\sigma}g^{\lambda\tau}R^\mu{}_{\nu\rho\sigma}A_\tau = R^\mu{}_{\nu\rho}{}^\nu A_\mu.$$

3 The Levi-Civita connection

The Levi-Civita connection is a particular connection associated with the metric. Its coefficients are called the *Christoffel symbols*, and read

$$\Gamma^\mu{}_{\nu\rho} \equiv \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}\right), \tag{13}$$

where $g^{\mu\sigma}$ are the components of the inverse metric, such that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$.

Q1. Check that $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$.

Solution Q1

The set of the first two terms is symmetrised in ρ, ν, while the third one is symmetric because of the symmetry of the metric itself. A connection which satisfies this property of symmetry is said to have no torsion.

Q2. Show that this connection is *metric preserving*, i.e. $\nabla_{\boldsymbol{u}}\boldsymbol{g} = \boldsymbol{0}$ for any vector field $\boldsymbol{u}$.

Solution Q2

In order to show that $\nabla_{\boldsymbol{u}}\boldsymbol{g} = \boldsymbol{0}$ for any $\boldsymbol{u} \in \Gamma(\mathcal{M})$, it is sufficient to prove it for $\boldsymbol{u} = \boldsymbol{\partial}_\mu$. The action of ∇_μ on a tensor like $\boldsymbol{g}$ returns $\nabla_\mu\boldsymbol{g} = (\nabla_\mu g_{\nu\rho})\,\mathbf{d}x^\nu \otimes \mathbf{d}x^\rho$, with

$$\begin{aligned}
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\sigma{}_{\nu\mu}g_{\sigma\rho} - \Gamma^\sigma{}_{\rho\mu}g_{\nu\sigma} \\
&= \partial_\mu g_{\nu\rho} - \frac{1}{2}g^{\sigma\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\nu\mu}\right)g_{\sigma\rho} - \frac{1}{2}g^{\sigma\lambda}\left(\partial_\rho g_{\lambda\mu} + \partial_\mu g_{\lambda\rho} - \partial_\lambda g_{\rho\mu}\right)g_{\nu\sigma} \\
&= \partial_\mu g_{\nu\rho} - \frac{1}{2}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu}\right) - \frac{1}{2}\left(\partial_\rho g_{\nu\mu} + \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu}\right) \\
&= 0,
\end{aligned}$$

The Levi-Civita connection is thus metric preserving (or metric compatible).

Q3. Deduce from this that the metric can freely get in and out of Levi-Civita covariant derivatives, for example

$$\nabla_\rho \left(g_{\mu\nu} u^\mu v^\nu\right) = g_{\mu\nu} \nabla_\rho (u^\mu v^\nu). \tag{14}$$

Solution Q3

The various Leibnitz laws imply that covariant derivatives work like derivatives for any kind of tensor operation: tensor product, contraction, etc. For example

$$\nabla_\rho \left(g_{\mu\nu} u^\mu v^\nu\right) = (\nabla_\rho g_{\mu\nu}) u^\mu u^\nu + g_{\mu\nu} \nabla_\rho (u^\mu u^\nu).$$

The first term vanishes by virtue of the previous result. We conclude that in any such situation, when the metric is involved in a tensor calculation, it can get in and out of covariant derivatives.