Vectors, Forms, and Tensors

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1 Vectors, forms, and tensors

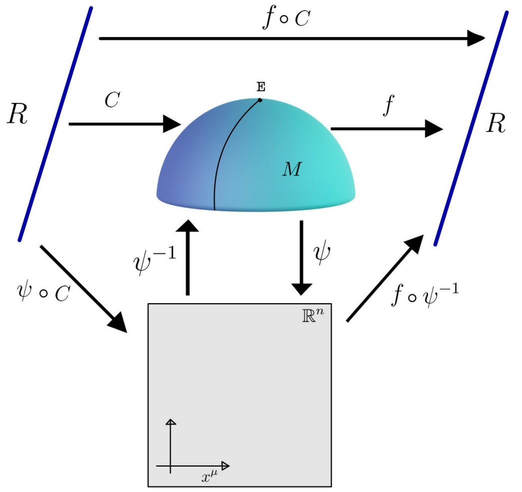
In this exercise, we review the definitions and properties of vectors, forms, and tensors in differential geometry. Let $\{x^\mu\}$ be a coordinate system over a Lorentzian manifold $\mathcal{M}$. We define the objects $\boldsymbol{\partial}_\mu$ by their action on any function $f : \mathcal{M} \to \mathbb{R}$ by

$$\boldsymbol{\partial}_\mu : f \mapsto \frac{\partial f}{\partial x^\mu}. \tag{1}$$

Q1. Show that, at each event E of $\mathcal{M}$, the set of objects $\boldsymbol{\partial}_\mu$ forms a basis of the tangent space $T_E\mathcal{M}$ of $\mathcal{M}$ at E. You will admit that $\dim T_E\mathcal{M} = \dim \mathcal{M}$ for simplicity.

Solution Q1

To begin with, we consider a smooth curve C on $\mathcal{M}$ such that $C : \mathbb{R} \to \mathcal{M}$. Assuming that t is a parameter along C, the goal is to describe the tangent vector d/dt in terms of ∂_μ at each point $E \in \mathcal{M}$ lying on the curve C. To do so, we probe the action of the tangent vector on an *arbitrary* function $f : \mathcal{M} \to \mathbb{R}$. In fact, in differential geometry, one fundamental way to define and understand tangent vectors at a specific point on a manifold is by looking at their action on functions.



$$\begin{aligned}\frac{d}{dt}f &= \frac{d}{dt}(f \circ C)\\ &= \frac{d}{dt}[(f \circ \psi^{-1}) \circ (\psi \circ C)]\\ &= \frac{d(\psi \circ C)^\mu}{dt}\frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\\ &= \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\\ &= \frac{dx^\mu}{dt}\partial_\mu f\end{aligned} \tag{2}$$

Given that the function f was chosen arbitrarily, we can write:

$$\frac{d}{dt} = \frac{dx^\mu}{dt}\partial_\mu \tag{3}$$

From the above equation it is evident that ∂_μ constitutes an appropriate basis for tangent vectors. Consequently, we can infer that they indeed form the basis for the tangent space at point E on

$\mathcal{M}$.

This means that any vector field on $\mathcal{M}$, $\boldsymbol{X} \in \Gamma(\mathcal{M})$, can be decomposed at every point, over this basis. This defines the *components* of $\boldsymbol{X}$ over the coordinate basis $\{\boldsymbol{\partial}_\mu\}$, with

$$\boldsymbol{X} = X^\mu \boldsymbol{\partial}_\mu. \tag{4}$$

Keep in mind that, rigorously speaking $\boldsymbol{X}, \boldsymbol{\partial}_\mu$ are vectors, while X^μ are simply functions $\mathcal{M} \to \mathbb{R}$. It is however customary to identify a vector with its components, as one would identify linear maps with matrices, and to abusively call the set of X^μ a vector.

Let $\{y^m\}$ be another coordinate system on $\mathcal{M}$, associated with its own set of vectors $\boldsymbol{\partial}_m$ associated with the partial derivatives with respect to y^m, similarly to eq. (1). The vector field $\boldsymbol{X}$ thus also enjoys a decomposition over $\{\boldsymbol{\partial}_m\}$ as

$$\boldsymbol{X} = X^m \boldsymbol{\partial}_m. \tag{5}$$

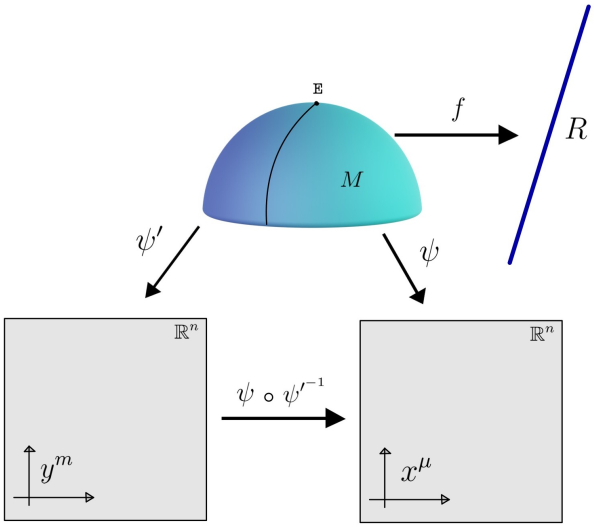
Q2. What is the relation between $\boldsymbol{\partial}_m$ and $\boldsymbol{\partial}_\mu$? And between the components X^μ and X^m?

Solution Q2

The relation between partial derivatives is given by the chain rule. For any function $f : \mathcal{M} \to \mathbb{R}$, considering that $f(y^m) = f[x^\mu(y^m)]$ we have

$$\boldsymbol{\partial}_m(f) = \frac{\partial}{\partial y^m} f[x^\mu(y^m)] = \frac{\partial x^\mu}{\partial y^m}\frac{\partial f}{\partial x^\mu} = \frac{\partial x^\mu}{\partial y^m}\boldsymbol{\partial}_\mu(f)$$

To delve deeper and provide more clarity, let's revisit the discussion using the notation of mappings; by definition we can write



$$\begin{aligned}\boldsymbol{\partial}_m(f) &= \frac{\partial}{\partial y^m} f \circ \psi'^{-1} \\ &= \frac{\partial}{\partial y^m}[(f \circ \psi^{-1}) \circ (\psi \circ \psi'^{-1})] \\ &= \frac{\partial(\psi \circ \psi'^{-1})^\mu}{\partial y^m}\frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \\ &= \frac{\partial x^\mu}{\partial y^m}\boldsymbol{\partial}_\mu(f)\end{aligned} \tag{6}$$

therefore:

$$\boldsymbol{\partial}_m = (\partial x^\mu/\partial y^m)\boldsymbol{\partial}_\mu$$

Besides, by comparing the decompositions of the vector $\boldsymbol{X}$ over both $\{\boldsymbol{\partial}_\mu\}$ and $\{\boldsymbol{\partial}_m\}$, we find

$$\boldsymbol{X} = X^m \boldsymbol{\partial}_m = X^m(\partial x^\mu/\partial y^m)\boldsymbol{\partial}_\mu,$$

whence $X^\mu = (\partial x^\mu/\partial y^m)X^m$.

A *differential form*, or *one-form*, $\boldsymbol{\omega} \in \Omega^1(\mathcal{M})$, is a field of linear forms. In other words, at each event E of $\mathcal{M}$, $\boldsymbol{\omega}$ eats a vector of the tangent space $T_E\mathcal{M}$, and returns a number. The set of all linear forms at an event E is called the cotangent space $T^*_E\mathcal{M}$ of $\mathcal{M}$ at E. Consider the set of differential forms denoted $\{\mathbf{d}x^\mu\}$, such that

$$\mathbf{d}x^\mu(\boldsymbol{\partial}_\nu) = \delta^\mu_\nu \equiv \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Q3. Show that for any vector field $\boldsymbol{X}$, $\mathbf{d}x^\mu(\boldsymbol{X}) = X^\mu$.

Solution Q3

Decomposing $\boldsymbol{X}$ over $\{\boldsymbol{\partial}_\mu\}$, and using that $\mathbf{d}x^\mu$ is linear, we find

$$\mathbf{d}x^\mu(\boldsymbol{X}) = \mathbf{d}x^\mu(X^\nu \boldsymbol{\partial}_\nu) = X^\nu \mathbf{d}x^\mu(\boldsymbol{\partial}_\nu) = X^\nu \delta^\mu_\nu = X^\mu.$$

Thus, $\mathbf{d}x^\mu$ can be seen as the application that selects the μth component of a vector.

Q4. Justify that, at each $E \in \mathcal{M}$, $\{\mathbf{d}x^\mu\}$ forms a basis of $T^*_E\mathcal{M}$.

Solution Q4

We already know that, at any event E, the $\mathbf{d}x^\mu$ are linear forms (they were defined this way). They are also linearly independent (meaning that no dx^μ can be written as a combination of the others), because if we consider a vanishing linear combination of them $\alpha_\mu \mathbf{d}x^\mu = \mathbf{0}$, i.e. such that

$$\forall \boldsymbol{X} \in T_E\mathcal{M} \qquad \alpha_\mu \mathbf{d}x^\mu(\boldsymbol{X}) = 0,$$

then it is clear that all the coefficients α_μ must be zero, as we can apply the above relation for each $\boldsymbol{X} = \boldsymbol{\partial}_\nu$. Put it more simply, If the dx^μ were not linearly independent, then some of them would be redundant, and we wouldn't actually need all of them to span the cotangent space. This would contradict the idea of them being a basis. Furthermore, it is a standard result that the dimension of the space of all linear form on a vector space has the same dimension as this vector space, hence $\dim(T_E\mathcal{M}) = \dim(T^*_E\mathcal{M}) = \dim \mathcal{M}$. The set of $\{\mathbf{d}x^\mu\}$ being linearly independent and having the right cardinal, it is a basis of $T^*_E\mathcal{M}$.

Any differential form $\boldsymbol{\omega}$ can thus be decomposed as $\boldsymbol{\omega} = \omega_\mu \mathbf{d}x^\mu$.

Q5. Show that $\omega_\mu = \boldsymbol{\omega}(\boldsymbol{\partial}_\mu)$.

Solution Q5

The decomposition of $\boldsymbol{\omega}$ over $\{\mathrm{d}x^\mu\}$ and the definition of that basis directly gives

$$\boldsymbol{\omega}(\boldsymbol{\partial}_\mu) = \omega_\nu \mathbf{d}x^\nu(\boldsymbol{\partial}_\mu) = \omega_\nu \delta^\mu_\nu = \omega_\mu.$$

Q6. If another coordinate system $\{y^m\}$ is used, we write $\boldsymbol{\omega} = \omega_m \mathbf{d}y^m$. What is the relation between ω_m and ω_μ? And between $\mathbf{d}y^m$ and $\mathbf{d}x^\mu$?

Solution Q6

If $\boldsymbol{\omega}$ is decomposed over the basis associated with another coordinate system $\{y^m\}$, then

$$\omega_m = \boldsymbol{\omega}(\boldsymbol{\partial}_m) = \boldsymbol{\omega}\left(\frac{\partial x^\mu}{\partial y^m}\boldsymbol{\partial}_\mu\right) = \frac{\partial x^\mu}{\partial y^m}\boldsymbol{\omega}(\boldsymbol{\partial}_\mu) = \frac{\partial x^\mu}{\partial y^m}\omega_\mu.$$

Now considering that $\boldsymbol{\omega} = \omega_\mu \mathbf{d}x^\mu = \omega_m \mathbf{d}y^m$, and using the above relation, we get directly

$$\omega_\mu \mathbf{d}x^\mu = \frac{\partial x^\mu}{\partial y^m}\omega_\mu \mathbf{d}y^m \rightarrow \mathbf{d}x^\mu = \frac{\partial x^\mu}{\partial y^m}\mathbf{d}y^m,$$

A *tensor field* $\boldsymbol{T}$ is a composite object, obtained by putting together vectors and forms; it belongs to the tensor product of some copies of $\Gamma(\mathcal{M})$ and $\Omega^1(\mathcal{M})$. For example, if

$$\boldsymbol{T} \in \Gamma(\mathcal{M}) \otimes \Omega^1(\mathcal{M}) \otimes \Omega^1(\mathcal{M}), \tag{8}$$

then $\boldsymbol{T}$ has the properties of a vector combined with two one-forms. In other words, $\boldsymbol{T}$ can be seen as a machine which eats two vectors and returns a vector:

$$\boldsymbol{T}: \begin{array}{r} \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}) \\ (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}). \end{array} \tag{9}$$

Q7. From the decompositions of vectors and forms, show that the general decomposition of the above tensor is

$$\boldsymbol{T} = T^{\mu}{}_{\nu\rho}\, \boldsymbol{\partial}_{\mu} \otimes \mathbf{d}x^{\nu} \otimes \mathbf{d}x^{\rho}. \tag{10}$$

Solution Q7

The tensor product $\otimes$ is just a way of putting vectors together. If V_1, V_2 are two vector spaces, $V_1 \otimes V_2$ is by definition the vector space made of all linear combinations of “juxtapositions” of the form $\boldsymbol{v}_1 \otimes \boldsymbol{v}_2$, where $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in V_1 \times V_2$. Note however the following crucial difference with the Cartesian product of vector spaces: while

$$\forall (\boldsymbol{v}_1, \boldsymbol{v}_2), (\boldsymbol{w}_1, \boldsymbol{w}_2) \in V_1 \times V_2 \qquad (\boldsymbol{v}_1, \boldsymbol{v}_2) + (\boldsymbol{w}_1, \boldsymbol{w}_2) = (\boldsymbol{v}_1 + \boldsymbol{w}_1, \boldsymbol{v}_2 + \boldsymbol{w}_2),$$

the tensor product is, on the contrary, *distributive* with respect to addition; in other words

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{v}_1 \otimes \boldsymbol{w}_2 = \boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 + \boldsymbol{w}_2),$$

but

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{w}_1 \otimes \boldsymbol{w}_2 \neq (\boldsymbol{v}_1 + \boldsymbol{w}_1) \otimes (\boldsymbol{v}_2 + \boldsymbol{w}_2).$$

If $\{\boldsymbol{e}_\alpha\}$ and $\{\boldsymbol{u}_\beta\}$ are respectively bases V_1 and V_2, then a basis of $V_1 \times V_2$ is simply $\{(\boldsymbol{e}_\alpha, \mathbf{0})\} \cup \{(\mathbf{0}, \boldsymbol{u}_\beta)\}$, so that $\dim(V_1 \times V_2) = \dim V_1 + \dim V_2$. On the contrary, a basis of $V_1 \otimes V_2$ is $\{\boldsymbol{e}_\alpha \otimes \boldsymbol{u}_\beta\}$, so that $\dim(V_1 \otimes V_2) = \dim V_1 \times \dim V_2$.

Once the above is understood, it is easy to deduce that a basis of $\Gamma(\mathcal{M}) \otimes \Omega^1(\mathcal{M}) \otimes \Omega^1(\mathcal{M})$ is the set of all possible combinations $\boldsymbol{\partial}_\mu \otimes \mathbf{d}x^\nu \otimes \mathbf{d}x^\rho$, so that any tensor $\boldsymbol{T}$ of this space is decomposed as

$$\boldsymbol{T} = T^{\mu}{}_{\nu\rho}\, \boldsymbol{\partial}_{\mu} \otimes \mathbf{d}x^{\nu} \otimes \mathbf{d}x^{\rho}.$$

Q8. How do the components $T^{\mu}{}_{\nu\rho}$ transform under a coordinate transformation $\{x^\mu\} \to \{y^m\}$?

Solution Q8

From the relation between $\boldsymbol{\partial}_\mu$, $\boldsymbol{\partial}_m$ and $\mathbf{d}x^\mu$, $\mathbf{d}y^m$, we get

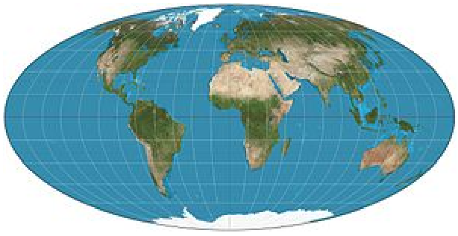
$$\begin{aligned} \boldsymbol{T} &= T^{\mu}{}_{\nu\rho}\, \boldsymbol{\partial}_{\mu} \otimes \mathbf{d}x^{\nu} \otimes \mathbf{d}x^{\rho} \\ &= T^{\mu}{}_{\nu\rho} \left(\frac{\partial y^m}{\partial x^\mu} \boldsymbol{\partial}_m \right) \otimes \left(\frac{\partial x^\nu}{\partial y^n} \mathbf{d}y^n \right) \otimes \left(\frac{\partial x^\rho}{\partial y^r} \mathbf{d}y^r \right) \\ &= \underbrace{\left(T^{\mu}{}_{\nu\rho} \frac{\partial y^m}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^n} \frac{\partial x^\rho}{\partial y^r} \right)}_{\equiv T^{m}{}_{nr}} \boldsymbol{\partial}_m \otimes \mathbf{d}y^n \otimes \mathbf{d}y^r, \end{aligned}$$

so

$$T^{m}{}_{np} = \frac{\partial y^m}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^n} \frac{\partial x^\rho}{\partial y^p} T^{\mu}{}_{\nu\rho}$$

and a similar inverse relation.

2 Mollweide projection of the sphere (continued)



The Mollweide projection is a particular coordinate system which allows one to represent the surface of a sphere on a flat map, where it appears as the interior of an ellipse. Contrary to the Mercator projection (used for most maps of the world), the Mollweide projection better preserves areas but changes angles.

If the sphere has a radius R, a point P on its surface with spherical coordinates $(\theta, \varphi) \in [0, \pi]\times]-\pi, \pi]$ has Mollweide coordinates defined by

$$x = \frac{2\sqrt{2}R}{\pi}\,\varphi\cos\psi \tag{11}$$

$$y = \sqrt{2}R\,\sin\psi, \tag{12}$$

where $\psi \in [-\pi/2, \pi/2]$ is an auxiliary angle implicitly defined by $2\psi + \sin 2\psi = \pi\cos\theta$.

Consider the operator $\boldsymbol{R}$ which rotates any vector tangent to the sphere by an angle δ clockwise. That is the linear map

$$\boldsymbol{R} : \boldsymbol{v} \mapsto \boldsymbol{R}(\boldsymbol{v}) = \boldsymbol{w}, \qquad \text{with} \qquad \begin{bmatrix} w^\theta \\ w^\varphi \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\theta\sin\delta \\ -\frac{\sin\delta}{\sin\theta} & \cos\delta \end{bmatrix} \begin{bmatrix} v^\theta \\ v^\varphi \end{bmatrix}. \tag{13}$$

The unusual $\sin\theta$ factors in the above rotation matrix are here to correct for the fact that $\boldsymbol{\partial}_\varphi$ is not a unit vector, but has length $\sin\theta$.

Q1. Justify that $\boldsymbol{R}$ is a tensor. What is the relation between its components and the matrix given in eq. (13)?

Solution Q1

The object $\boldsymbol{R}$ is a field of linear maps, which at each point of the sphere eats a vector and returns a vector. Thus it belongs to the tensor product of differential forms and vectors. We choose to write it as

$$\boldsymbol{R} = R^\alpha{}_\beta\, \boldsymbol{\partial}_\alpha \otimes \mathbf{d}x^\beta.$$

Note that $\boldsymbol{R} = R_\alpha{}^\beta\, \mathbf{d}x^\alpha \otimes \boldsymbol{\partial}_\beta$ would have been another possible choice. The action of $\boldsymbol{R}$ on a vector is then given by

$$\boldsymbol{w} = \boldsymbol{R}(\boldsymbol{v}) = (R^\alpha{}_\beta\, \boldsymbol{\partial}_\alpha)\, \mathbf{d}x^\beta(\boldsymbol{v}) = (R^\alpha{}_\beta v^\beta)\, \boldsymbol{\partial}_\alpha$$

so that $w^\alpha = R^\alpha{}_\beta v^\beta$. We therefore recognise in $R^\alpha{}_\beta$ the components of the matrix given in eq. (13), where index α labels the rows, while index β labels the columns.

Q2. Determine the matrix of $\boldsymbol{R}$ in the Mollweide coordinate system.

Solution Q2

The components of $\boldsymbol{R}$ with respect to Mollweide coordinates are given by the general transformation rule

$$R^a{}_b = \frac{\partial x^a}{\partial\theta^\alpha}\frac{\partial\theta^\beta}{\partial x^b} R^\alpha{}_\beta$$

Remember from last time we calculated

$$\begin{aligned}
\frac{\partial x}{\partial \theta} &= -\frac{2\sqrt{2}R}{\pi}\varphi\frac{\partial \psi}{\partial \theta}\sin\psi = -x\frac{\partial \psi}{\partial \theta}\tan\psi \\
\frac{\partial x}{\partial \varphi} &= \frac{2\sqrt{2}R}{\pi}\cos\psi = \frac{2}{\pi}\sqrt{2R^2-y^2} \\
\frac{\partial y}{\partial \theta} &= \sqrt{2}R\frac{\partial \psi}{\partial \theta}\cos\psi = \frac{\partial \psi}{\partial \theta}\sqrt{2R^2-y^2} \\
\frac{\partial y}{\partial \varphi} &= 0.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \psi}{\partial \theta} &= -\frac{\pi}{2}\frac{\sin\theta}{1+\cos 2\psi} \\
&= -\frac{\pi}{4}\frac{\sqrt{1-\left(\frac{2\psi+\sin 2\psi}{\pi}\right)^2}}{1-\sin^2\psi} \\
&= -\frac{\pi}{4}\left(1-\frac{y^2}{2R^2}\right)^{-1}\sqrt{1-\frac{4}{\pi^2}\left[\arcsin\left(\frac{y}{\sqrt{2}R}\right)+\frac{y}{\sqrt{2}R}\sqrt{1-\frac{y^2}{2R^2}}\right]^2} \\
&\equiv F(y).
\end{aligned}$$

Representing these using matrices we can write

$$\left[\frac{\partial x^a}{\partial \theta^\alpha}\right] = \begin{bmatrix} -\frac{xyF(y)}{\sqrt{2R^2-y^2}} & \frac{2}{\pi}\sqrt{2R^2-y^2} \\ \sqrt{2R^2-y^2}F(y) & 0 \end{bmatrix}$$

and hence its inverse is

$$\left[\frac{\partial \theta^\beta}{\partial x^b}\right] = \begin{bmatrix} 0 & \frac{1}{F\sqrt{2R^2-y^2}} \\ \frac{\pi}{2\sqrt{2R^2-y^2}} & \frac{\pi xy}{2(2R^2-y^2)^{3/2}} \end{bmatrix}$$

so that finally

$$\begin{aligned}
[R^a{}_b] &= \begin{bmatrix} -\frac{xyF(y)}{\sqrt{2R^2-y^2}} & \frac{2}{\pi}\sqrt{2R^2-y^2} \\ \sqrt{2R^2-y^2}F(y) & 0 \end{bmatrix}\begin{bmatrix} \cos\delta & \sin\theta\sin\delta \\ -\frac{\sin\delta}{\sin\theta} & \cos\delta \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{F\sqrt{2R^2-y^2}} \\ \frac{\pi}{2\sqrt{2R^2-y^2}} & \frac{\pi xy}{2(2R^2-y^2)^{3/2}} \end{bmatrix} \\
&= \begin{bmatrix} \cos\delta + \frac{xyF^2}{2R^2-y^2}\sin\delta & \frac{2R^2\sin\delta}{F^2(2R^2-y^2)}\left[1+F^4\left(\frac{xy}{2R^2}\right)^2\right] \\ -\frac{2R^2F^2\sin\delta}{2R^2-y^2} & \cos\delta - \frac{xyF^2}{2R^2}\sin\delta \end{bmatrix}.
\end{aligned}$$

3 Tensor: Some practice

This is a computational exercise about the notions of vectors, forms and tensors. You can skip it if you feel comfortable enough.
We consider the Cartesian coordinates $x^\mu = (t,x,y,z)$ and the Polar coordinates $\tilde{x}^\mu = (t,r,\phi,z)$. They are related as

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan\left(\frac{y}{x}\right) \end{cases}.$$

We consider the metric

$$\boldsymbol{g} = -\mathbf{d}t^2 + f(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)\,,$$

and the 4-vector

$$\boldsymbol{u} = \gamma \left(\boldsymbol{\partial}_t + \frac{v}{\sqrt{f(t)}} \boldsymbol{\partial}_x \right)\,,$$

where $f(t) > 0$ is a monotonous smooth function, $v < 1$ and $\gamma = (1 - v^2)^{-1/2}$.

Q1. Compute u_μ, $g^{\mu\nu}$, $h_{\mu\nu}$, $h_\mu{}^\nu$ and $h^{\mu\nu}$ where the tensor $\boldsymbol{h}$ is defined as

$$\boldsymbol{h} = \boldsymbol{g} + \boldsymbol{u} \otimes \boldsymbol{u}\,,$$

and

$$(\boldsymbol{u} \otimes \boldsymbol{u})_{\mu\nu} = u_\mu u_\nu\,.$$

Solution Q1

Note: In this solution and in the following, we will freely go from the matrix notation to the full tensorial notation (with the partial derivatives and the differentials). This is a slight abuse of notation but it should be clear form the context. It is important to be comfortable with both notations.

We have:

$$\begin{aligned} \boldsymbol{g} &= g_{\mu\nu} \mathbf{d}x^\mu \mathbf{d}x^\nu \\ \boldsymbol{u} &= u^\nu \boldsymbol{\partial}_\nu \end{aligned} \tag{14}$$

We use the metric to compute the covariant form

$$u_\mu = g_{\mu\nu} u^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \frac{v}{\sqrt{f}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\gamma \\ \gamma v \sqrt{f} \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

The tensor $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, which is defined such that $g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu$. As the metric is diagonal, the inverse is computed directly, which yields

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & f^{-1} & 0 \\ 0 & 0 & 0 & f^{-1} \end{pmatrix}\,. \tag{16}$$

To compute the tensor $\boldsymbol{h}$ we first compute

$$\begin{aligned} (\boldsymbol{u} \otimes \boldsymbol{u})_{\mu\nu} = u_\mu u_\nu &= \begin{pmatrix} -\gamma \\ \gamma v \sqrt{f(t)} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\gamma & \gamma v \sqrt{f(t)} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 & -\gamma^2 v \sqrt{f(t)} & 0 & 0 \\ -\gamma^2 v \sqrt{f(t)} & \gamma^2 v^2 f(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

which leads to

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu = \begin{pmatrix} -1 + \gamma^2 & -v\gamma^2 \sqrt{f} & 0 & 0 \\ -v\gamma^2 \sqrt{f} & \gamma^2 f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix}\,.$$

To compute ${h_\mu}^\nu$, we have:

$${h_\mu}^\nu = g^{\lambda\nu} h_{\mu\lambda}$$

Performing this multiplication, we get

$${h_\mu}^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{f(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{f(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{f(t)} \end{pmatrix} \times \begin{pmatrix} -1+\gamma^2 & -v\gamma^2\sqrt{f} & 0 & 0 \\ -v\gamma^2\sqrt{f} & \gamma^2 f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 1-\gamma^2 & v\gamma^2\sqrt{f} & 0 & 0 \\ v\gamma^2\sqrt{f} & \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, we can compute $h^{\mu\nu}$ in the same way as

$$h^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} h_{\lambda\rho} = g^{\mu\lambda} {h_\lambda}^\nu$$

which gives

$$h^{\mu\nu} = \begin{pmatrix} \gamma^2 - 1 & \frac{v\gamma^2}{\sqrt{f}} & 0 & 0 \\ \frac{v\gamma^2}{\sqrt{f}} & \frac{\gamma^2}{f} & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix} .$$

Q2. Compute $u^2 \equiv u^\mu u^\nu g_{\mu\nu}$ and $h_{\mu\nu} u^\mu$.

Solution Q2

The norm squared of the 4-velocity is

$$u^2 = u^\mu g_{\mu\nu} u^\nu = u^\mu u_\mu = -1 \,, \tag{17}$$

which shows that the affine parameter has been chosen to normalise the 4-velocity.
To compute $h_{\mu\nu} u^\mu$ one can use the matrix notation, but a direct method yields

$$h_{\mu\nu} u^\mu = (g_{\mu\nu} + u_\mu u_\nu) u^\mu = u_\nu + u_\mu u^\mu u_\nu = u_\nu + (-1) u_\nu = 0 \,. \tag{18}$$

We consider the function

$$\alpha(t,x,y,z) = e^{H_0 t} \cos(kx)$$

where H_0 and k have unites of inverse time, and the vector

$$\boldsymbol{r} = t\boldsymbol{\partial}_t + y\boldsymbol{\partial}_y \,.$$

Q3. Compute the form $\omega = \mathbf{d}\alpha$ in the Cartesian basis.

Solution Q3

The differential form $\mathbf{d}\alpha$ can be expanded as:

$$\omega = \frac{\partial\alpha}{\partial t}\mathbf{d}t + \frac{\partial\alpha}{\partial x}\mathbf{d}x + \frac{\partial\alpha}{\partial y}\mathbf{d}y + \frac{\partial\alpha}{\partial z}\mathbf{d}z$$

therefore we get

$$\omega = e^{H_0 t} \left(H_0 \cos(kx) \mathbf{d}t - k \sin(kx) \mathbf{d}x \right) \,. \tag{19}$$

Q4. Compute the scalar product $r^\mu\omega_\mu$ and check that $r^\mu\omega_\mu = \boldsymbol{r}(f)$.

Solution Q4

In vector notation, one has

$$r^\mu = (t, 0, y, 0)\,,$$

which implies

$$r^\mu\omega_\mu = H_0 t e^{H_0 t}\cos(kx)\,. \tag{20}$$

On the other side, applying the vector to the function $f = \alpha(t, x, y, z)$ gives

$$\boldsymbol{r}(f) = t\partial_t f + y\partial_y f = H_0 t e^{H_0 t}\cos(kx)\,, \tag{21}$$

which is the same expression as $r^\mu\omega_\mu$

Q5. Compute the components of the metric $\boldsymbol{g}$ and of the 4-vector $\boldsymbol{u}$ in the Polar coordinates.

Solution Q5

To compute the change of coordinates, there are two different ways. Fisrt, one can use the definitions with the partial derivatives (beware of the indices) ! Another method is to write the tensors explicitely using the $\boldsymbol{\partial}\mu$ and $\mathrm{d}x^\mu$ and their transformations under a change of coordinates. Both methods should give the same results. We have

$$\begin{cases} \mathbf{d}t = \mathbf{d}t \\ \mathbf{d}x = \cos(\phi)\mathbf{d}r - r\sin(\phi)\mathbf{d}\phi \\ \mathbf{d}y = \sin(\phi)\mathbf{d}r + r\cos(\phi)\mathbf{d}\phi \\ \mathbf{d}z = \mathbf{d}z \end{cases}$$

Substituting these in the Cartesian form of the metric we get

$$\boldsymbol{g} = -\mathbf{d}t^2 + f(t)\,\mathbf{d}r^2 + r^2 f(t)\,\mathbf{d}\phi^2 + f(t)\,\mathbf{d}z^2\,, \tag{22}$$

Using the chain rule we can express the basis vectors in Cartesian coordinates in terms of the basis vectors in Polar coordinate

$$\boldsymbol{\partial}_x = \frac{\partial r}{\partial x}\boldsymbol{\partial}_r + \frac{\partial \phi}{\partial x}\boldsymbol{\partial}_\phi$$

and knowing that

$$r = \sqrt{x^2 + y^2}$$
$$\tan(\phi) = \frac{y}{x}$$

we find

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos(\phi)$$
$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin(\phi)}{r}$$

Now, substituting these into our relations for the basis vectors, we get

$$\boldsymbol{\partial}_x = \cos(\phi)\boldsymbol{\partial}_r - \frac{\sin(\phi)}{r}\boldsymbol{\partial}_\phi$$

$$\boldsymbol{\partial}_t = \boldsymbol{\partial}_t$$

whence

$$\boldsymbol{u} = \gamma \left(\boldsymbol{\partial}_t + \frac{v}{\sqrt{f}} \cos(\phi) \boldsymbol{\partial}_r - \frac{v}{r\sqrt{f}} \sin(\phi) \boldsymbol{\partial}_\phi \right) . \tag{23}$$