

The Schwarzschild Solution

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1 Gargantua

This exercise discusses certain aspects of the black hole *Gargantua* in the movie *Interstellar* by C. Nolan. We will neglect the spin of the black hole and assume that it can be described by the Schwarzschild geometry.

A particularly impressive moment of the movie is the visit of a planet (*Miller's planet*), orbiting around Gargantua, where one hour correspond to seven years for an observer far away from the black hole. We will assume for simplicity that the trajectory of the planet is circular.

Q1. How close should Miller's planet be to the Schwarzschild radius of the black hole? Is that actually possible?

Solution Q1

For a circular orbit in the equatorial plane ($\theta = \frac{\pi}{2}$) we have $dr = d\theta = 0$; so the metric simplifies to:

$$d\tau^2 = A(r)dt^2 - r^2d\varphi^2.$$

dividing both sides by $d\tau^2$ we get

$$\left(1 - \frac{r_S}{r}\right) \dot{t}^2 = 1 + \frac{L^2}{r^2} = 1 + v^2$$

where in the last step we have use the relation $L = \frac{I\Omega}{m} = r^2\Omega = rv_{\perp}$. As v represents the physical velocity of the orbiting planet, it cannot be bigger than 1. Hence in order to have a time dilation $\dot{t} = 7 \text{ y}/1 \text{ h} = 6 \times 10^4$, the planet should be essentially at the horizon of Gargantua, with

$$\frac{r - r_S}{r_S} \approx 1.6 \times 10^{-5} \quad (1)$$

This is impossible for a Schwarzschild black hole, as we know that the closest stable orbit is such that $r = 3r_S$. This means that we cannot actually neglect the spin of the black hole: Gargantua is a *Kerr* black hole.

Let us overlook this issue, and turn to tidal forces. In the movie, Gargantua is supposed to be a supermassive black hole of mass $M = 10^8 M_{\odot}$.

Q2. Calculate the tidal effect of Gargantua on Miller's planet, assuming that the latter is comparable to the Earth. Estimate the height of tides from a simple model, and compare to the 1.2 km given in the movie.

Solution Q2

For a simple model, we assume Millers planet to be completely covered by water. As shown for example in John R. Taylor "Classical Mechanics" (pg. 333), the water surface is a surface of constant potential. Thus, we will compute the maximum tidal potential difference, which is compensated by a different water level.

First, we use the geodesic deviation equation to derive the relative force acting on test particles located at a separation δx^μ from the center of Millers planet compared to the force acting on the center of the planet ¹.

$$a^\mu = D_\tau^2 \delta x^\mu = -R_{\nu\lambda\rho}^\mu \dot{x}^\nu \dot{x}^\rho \delta x^\lambda . \quad (2)$$

Using $\dot{r} = 0$ and assuming $L/r \approx 1$, we get $\dot{t}^2 = 2r/(r - r_s)$ and $R_{trt}^r = -2R_{t\phi t}^\phi = r_s(r_s - r)/r^4$.

Proof of : $R_{trt}^r = \frac{r_s(r_s - r)}{r^4}$

non-vanishing Christoffel symbols in Schwarzschild solution:

$$\begin{aligned} \Gamma_{tt}^r &= \frac{GM}{r^3}(r - 2GM), & \Gamma_{rr}^r &= \frac{-GM}{r(r - 2GM)}, & \Gamma_{tr}^t &= \frac{GM}{r(r - 2GM)} \\ \Gamma_{r\theta}^\theta &= \frac{1}{r}, & \Gamma_{\theta\theta}^r &= -(r - 2GM), & \Gamma_{r\phi}^\phi &= \frac{1}{r} \\ \Gamma_{\phi\phi}^r &= -(r - 2GM) \sin^2 \theta, & \Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta, & \Gamma_{\theta\phi}^\phi &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

Therefore we can write

$$\begin{aligned} R_{trt}^r &= \partial_r \Gamma_{tt}^r - \partial_t \Gamma_{rt}^r + \Gamma_{r\lambda}^r \Gamma_{tt}^\lambda - \Gamma_{t\lambda}^r \Gamma_{rt}^\lambda \\ &= \partial_r \Gamma_{tt}^r + \Gamma_{rr}^r \Gamma_{tt}^r - \Gamma_{tt}^r \Gamma_{rt}^t \end{aligned}$$

Replacing the Christoffel symbols we get

$$\begin{aligned} R_{trt}^r &= \partial_r \left(\frac{GM}{r^3}(r - 2GM) \right) + \frac{-GM}{r(r - 2GM)} \cdot \frac{GM}{r^3}(r - 2GM) - \frac{GM}{r^3}(r - 2GM) \cdot \frac{GM}{r(r - 2GM)} \\ &= -3 \frac{GM}{r^4}(r - 2GM) + \frac{GM}{r^3} - 2 \frac{G^2 M^2}{r^4} \\ &= \frac{4G^2 M^2 - 2GM r}{r^4} \\ &= \frac{r_s(r_s - r)}{r^4} \end{aligned}$$

Thus we can write the relevant geodesic deviation equations as

$$a^r = D_\tau^2 \delta x^r = -R_{trt}^r (\dot{t})^2 \delta x^r = \frac{2r_s}{r^3} \delta x^r \quad (3)$$

$$a^\phi = D_\tau^2 \delta x^\phi = -R_{t\phi t}^\phi (\dot{t})^2 \delta x^\phi = -\frac{r_s}{r^3} \delta x^\phi . \quad (4)$$

$$(5)$$

Let us call the point of Millers planet that faces Gargantua the north pole and the point farthest away from Gargantua the south pole.

The relative force towards the center of Millers planet of a test particle (with unit mass) at a point between the planets center and the north pole at a distance R is

$$f_N(R) = -\frac{2}{r_s^2} R \quad (6)$$

¹The geodesic deviation equation describes how δx^μ (the separation vector) changes as we move along the geodesics.

and between equator and center, it is

$$f_E(R) = \frac{1}{r_s^2} R, \quad (7)$$

where we approximate $r \approx r_s$. The tidal potential difference at the surface of the planet is

$$\Delta U^t = U_E^t - U_N^t = \int_0^{R_M} dR f_E(R) - \int_0^{R_M+h} dR f_N(R) = \frac{3R_M^2}{2r_s^2} + \frac{h^2}{r_s^2} + \frac{2hR_M}{r_s^2}. \quad (8)$$

We assume that the water level at the north pole will be higher by h . The resulting potential difference due to Millers planet is

$$\Delta U^M = -\frac{GM_M}{R_M+h} + \frac{GM_M}{R_M} = \frac{G \cdot M_M \cdot h}{R_M \cdot (R_M+h)} = \frac{g \cdot R_M^2 \cdot h}{R_M \cdot (R_M+h)} = \frac{g \cdot h}{(1+h/R_M)}. \quad (9)$$

where M_M is the mass of Millers planet. We take $g \approx 10 \text{ m/s}^2$. Since we look for a equipotential surface, we can state

$$\Delta U^M = \Delta U^t \quad (10)$$

and solve for h . However, in this case, the equation cannot be solved meaningfully, because the tidal forces are too large. If we assume a small $h \ll R_M$, we can rewrite equation (8) to

$$\Delta U^t = U_E^t - U_N^t = \int_0^{R_M} dR f_E(R) - \int_0^{R_M} dR f_N(R) = \frac{3R_M^2}{2r_s^2} + O(h). \quad (11)$$

Equating $\Delta U^t = \Delta U^M = g \cdot h + O(h^2)$ will lead to (including c^2 to give correct units)

$$h = \frac{3c^2}{2gr_s^2} R_M^2 \approx 24000 \text{ km}, \quad (12)$$

invalidating our assumption of a small h . Unfortunately, the simple model does not work for our case. Applying our results to an earth - moon scenario, we would find

$$\Delta U^t = U_E^t - U_N^t = \frac{3M_m GR_E^2}{r_{Em}^3} + O(h), \quad (13)$$

where M_m , R_E , r_{Em} are the mass of the moon, the radius of the earth and the distance between earth and moon respectively. The resulting height of the tide is

$$h = \frac{3M_m GR_E^2}{g r_{Em}^3} \approx 106 \text{ cm} \quad (14)$$

Q3. In fact, there should not be tides on this planet. Any guess why?

Hint: If the Moon had oceans, would there be tides due to the Earth on it?

Solution Q3

The Moon is said to be “tidally locked” to the Earth: its own rotation has exactly the same angular velocity as its revolution, which results into the fact that we always see the same face of the Moon, while the opposite face is always hidden. This has not been always the case, it is an equilibrium situation that was reached during the course of the history of the Solar system. The main mechanism at work is that the Moon itself is not completely rigid; thus, it gets slightly distorted by the tidal forces created by the Earth. When the Moon’s rotation was not synchronised with its revolution, the axis of tidal distortion would rotate in the Moon’s frame, causing friction in the internal structure of the Moon. This friction dissipates energy, and eventually tends to synchronise rotation and revolution. This process is at work in any astronomical system, and will eventually happen to Earth with the Sun. Tidal locking is quicker if the tidal field is stronger, and hence it very rapidly occurs in a situation like Miller’s planet with Gargantua. In a tidally locked planet, there cannot be oceanic tides, because it is always the same point on the ground which undergoes high tides or low tides, at any moment.

2 Black hole shadow

Consider the Schwarzschild metric of a spherically symmetric black hole,

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - r_s/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (15)$$

Q1. Show that a photon moving in the equatorial plane $\theta = \pi/2$ follows a trajectory $r(\phi)$ obeying

$$\left(\frac{dr}{d\phi}\right)^2 + V(r) = 0, \quad (16)$$

and determine $V(r)$.

Solution Q1

Recall that (affinely parametrized) geodesics can be obtained as the equations of motion of the Lagrangian

$$L(x^\mu, \dot{x}^\mu) = g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu, \quad (17)$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$, with λ an affine parameter along the null geodesic.

We are interested in motion in the plane $\theta = \pi/2$ of the Schwarzschild geometry. The metric restricted to this plane leads to the Lagrangian

$$L(x^\mu, \dot{x}^\mu) = - \left(1 - \frac{r_S}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{1 - r_S/r} + r^2 \dot{\phi}^2. \quad (18)$$

Since t and ϕ are cyclic coordinates, we obtain two conserved quantities:

$$\frac{\partial L}{\partial \dot{t}} = \text{const} \quad \Rightarrow \quad \left(1 - \frac{r_S}{r}\right) \dot{t} \equiv E = \text{const}, \quad (19)$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const} \quad \Rightarrow \quad r^2 \dot{\phi} \equiv L = \text{const}. \quad (20)$$

In the asymptotic region $r \rightarrow \infty$ the metric becomes Minkowski and these constants can be interpreted as the energy (E) and angular momentum (L) of the photon.

For null geodesics we must also impose

$$0 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad (21)$$

which using the conserved charges becomes

$$0 = - \frac{E^2}{1 - r_S/r} + \frac{\dot{r}^2}{1 - r_S/r} + \frac{L^2}{r^2}. \quad (22)$$

Using the chain rule,

$$\dot{r} = \frac{dr}{d\lambda} = \frac{dr}{d\phi} \frac{d\phi}{d\lambda} = \frac{dr}{d\phi} \frac{L}{r^2}, \quad (23)$$

we plug this into the previous equation to obtain

$$\left(\frac{dr}{d\phi}\right)^2 - r^4 \frac{E^2}{L^2} + r^2 \left(1 - \frac{r_S}{r}\right) = 0. \quad (24)$$

Thus, we can write

$$\left(\frac{dr}{d\phi}\right)^2 + V(r) = 0, \quad V(r) = r^2 \left(1 - \frac{r_S}{r} - r^2 \frac{E^2}{L^2}\right). \quad (25)$$

Q2. Show that $V(r)$ can be expressed solely in terms of r , r_s and the impact parameter b .

Hint: You may study (16) at large r to relate b to other constants of motion.

Solution Q2

In the asymptotic region $r \rightarrow \infty$, the constants E and L are the energy and angular momentum of the photon. The angular momentum of a photon is

$$\vec{L} = \vec{r} \times \vec{p} = rp \sin \alpha \hat{e}_z, \quad (26)$$

where p is the magnitude of the spatial momentum and α is the angle between \vec{r} and \vec{p} . For a photon, $p = E$. The impact parameter b is defined as

$$b = r \sin \alpha, \quad (27)$$

so that

$$L = bE \quad \iff \quad b = \frac{L}{E}. \quad (28)$$

Hence we may write

$$V(r) = r^2 \left(1 - \frac{r_S}{r} - \frac{r^2}{b^2} \right), \quad (29)$$

and the radial equation as

$$\left(\frac{dr}{d\phi} \right)^2 + V(r) = 0, \quad V(r) = r^2 \left(1 - \frac{r_S}{r} - \frac{r^2}{b^2} \right). \quad (30)$$

Another way to see that $b = L/E$ is to consider the limit $r \rightarrow \infty$ in the equation of motion. Neglecting r_S/r in that limit, one can check that the straight-line trajectory

$$b = r \sin \phi \quad (31)$$

solves the equation provided $b = L/E$.

Q3. Photons sent from infinity towards the black hole may or may not fall through the horizon depending on the impact parameter b . Compute the critical impact parameter b_c such that all photons with $b < b_c$ fall through the horizon.

Solution Q3

The equation of motion can be interpreted as that of a one-dimensional particle of zero total energy moving in the potential $V(r)$. If $V(r)$ is positive in some interval between $r = \infty$ and $r = r_S$, the “particle” (photon) does not have enough energy to cross the potential barrier and fall into the black hole.

We therefore look for the extrema of $V(r)$:

$$V'(r) = -r_S + 2r - \frac{4r^3}{b^2}. \quad (32)$$

Extrema occur at $r = r_*$ such that

$$-r_S + 2r_* - \frac{4r_*^3}{b^2} = 0 \quad \iff \quad \frac{r_*^3}{b^2} = \frac{1}{4}(2r_* - r_S). \quad (33)$$

The critical impact parameter b_c is the value for which the maximum of the potential touches zero, i.e. $V(r_*) = 0$. Thus r_* must solve

$$0 = V(r_*) = r_*^2 \left(1 - \frac{r_S}{r_*} - \frac{r_*^2}{b_c^2} \right). \quad (34)$$

Using the relation for r_*^3/b_c^2 from above, this becomes

$$0 = 1 - \frac{r_S}{r_*} - \frac{r_*^2}{b_c^2} \quad (35)$$

$$= 1 - \frac{r_S}{r_*} - \left(\frac{1}{4}(2r_* - r_S) \right) \frac{1}{r_*} \quad (36)$$

$$= 1 - \frac{r_S}{r_*} - \frac{1}{4} \left(2 - \frac{r_S}{r_*} \right) \quad (37)$$

$$= \frac{1}{4} \left(2 - 3 \frac{r_S}{r_*} \right). \quad (38)$$

Hence,

$$r_*(b_c) = \frac{3}{2} r_S. \quad (39)$$

Substituting this back into

$$-r_S + 2r_* - \frac{4r_*^3}{b_c^2} = 0 \quad (40)$$

gives

$$2r_S - \frac{27}{2} \frac{r_S^3}{b_c^2} = 0 \implies b_c = \frac{3\sqrt{3}}{2} r_S. \quad (41)$$

Q4. What is the impact parameter b_1 such that the incoming photon goes around the black hole once and then continues exactly in the same direction? You may define b_1/r_s implicitly in a form that could easily be given to a computer (for example, as a solution of an equation, possibly involving integrals).

Solution Q4

We now want to express $\phi(r)$ and find the impact parameter b_1 such that the photon completes one full loop around the black hole and then escapes in exactly the same direction, i.e. the total deflection angle is

$$\Delta\phi = 3\pi. \quad (42)$$

From the radial equation

$$\left(\frac{dr}{d\phi} \right)^2 = -V(r) = r^2 \left(\frac{r^2}{b^2} + \frac{r_S}{r} - 1 \right), \quad (43)$$

we obtain

$$d\phi = -\frac{dr}{r} \frac{1}{\sqrt{r^2/b^2 + r_S/r - 1}}, \quad (44)$$

where we choose the minus sign because r decreases as ϕ increases along the incoming part of the trajectory.

The total deflection angle is twice the angle accumulated between $r = \infty$ and the point of closest approach $r = r_0$, where $dr/d\phi = 0$. At r_0 we have $V(r_0) = 0$, hence

$$1 - \frac{r_S}{r_0} - \frac{r_0^2}{b^2} = 0 \implies b = \frac{r_0}{\sqrt{1 - r_S/r_0}}. \quad (45)$$

Thus the total angle is

$$\Delta\phi = 2 \int_{r_0}^{\infty} \frac{dr}{r} \frac{1}{\sqrt{r^2/b^2 + r_S/r - 1}}. \quad (46)$$

We require

$$\Delta\phi = 3\pi. \quad (47)$$

To put the integral in dimensionless form, define $\rho = r/r_0$, so $dr = r_0 d\rho$. Using

$$\frac{r_0^2}{b^2} = 1 - \frac{r_S}{r_0}, \quad (48)$$

we obtain

$$\frac{3\pi}{2} = \int_{r_0}^{\infty} \frac{dr}{r} \frac{1}{\sqrt{r^2/b^2 + r_S/r - 1}} \quad (49)$$

$$= \int_1^{\infty} \frac{d\rho}{\rho} \frac{1}{\sqrt{\rho^2 - 1 + \frac{r_S}{r_0}(\frac{1}{\rho} - \rho^2)}} \quad (50)$$

$$\equiv F\left(\frac{r_S}{r_0}\right). \quad (51)$$

The function $F(x)$ can be evaluated numerically. One finds that the condition

$$F\left(\frac{r_S}{r_0}\right) = \frac{3\pi}{2} \quad (52)$$

is satisfied for

$$\frac{r_S}{r_0} \approx 0.644\dots \quad (53)$$

Then

$$b_1 = \frac{r_0}{\sqrt{1 - r_S/r_0}} \approx 2.602 r_S. \quad (54)$$

This b_1 is the impact parameter such that the photon executes one full loop around the black hole before emerging in the original direction.