The Schwarzschild Solution

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This sheet deals with the first exact solution of Einstein's equation, discovered by Karl Schwarzschild in December 1915, in the fires of World War I. This solution describes quite accurately the gravitational field created by the Sun, leading to tests of general relativity in the Solar System, but also the most intriguing objects in the Universe: black holes.

The Schwarzschild metric represents the spacetime geometry at the exterior of a static and spherically symmetric body with mass M. Its line element reads

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \qquad A(r) \equiv 1 - \frac{r_\mathrm{S}}{r}, \tag{1}$$

where $r_\mathrm{S} \equiv 2GM$ is the Schwarzschild radius of the central object, at $r = 0$. The coordinates (t, r, θ, φ) are sometimes called the Droste coordinates. It is good to keep in mind its order of magnitude for the Sun, namely $r_\mathrm{S}^\odot = 2GM_\odot \approx 3\,\mathrm{km}$.

1 Precession of Mercury's perihelion

Like all the planets of the Solar System, Mercury has an elliptic orbit around the Sun. This orbit is not entirely stationary: the axes of the ellipse tend to slowly rotate, with an angular velocity of 5600 arcsec/century. This is known as the precession of Mercury's perihelion. Most of this precession (5026 arcsec/century) is due to the fact that the Sun is not completely spherical, which affects the gravitational field it generates. There is also the effect of the other planets of the Solar System (mostly Venus, Jupiter, and the Earth), responsible for 531 arcsec/century.

Once those effects are taken into account, there are still 43 arcsec/century which are not explained by Newtonian physics. In this exercise, we are going to see that these are due to post-Newtonian corrections of general relativity.

Q1. Show that the equation of motion for a freely falling test particle imply

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[A(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = 0, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = 0, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\,\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right) = 0, \tag{4}$$

where τ is the particle's proper time.

HINT: Start from the length proper time $L = \int \sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}d\tau$, and use the Schwarzschild metric to express it in terms of $t(\tau), r(\tau), \theta(\tau), \varphi(\tau)$. Consider small variations of the worldline in just one coordinate at a time, e.g. $t(\tau)$ to $t(\tau) + \delta t(\tau)$ and impose $\delta L = 0$. Carry out this variational procedure for θ and φ as well, and show that the vanishing of δL leads to the three differential equations given in the problem.

Solution Q1

The geodesics of the test particles can be defined as a curve of extremal length, where the length is defined as

$$L = \int \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|} d\tau, \tag{5}$$

where a dot denotes a derivative with respect to the proper time τ. Here the integral is computed over an arbitrary path. The trajectory can be obtained imposing $\delta_\alpha L = 0$, where δ_α denotes a variation with respect to the following infinitesimal transformation:

$$\begin{cases} x^\beta(\tau) \to x^\beta(\tau) + \delta x^\beta(\tau) \text{ for } \beta = \alpha, \\ x^\beta(\tau) \to x^\beta(\tau) \text{ for } \beta \neq \alpha. \end{cases} \tag{6}$$

Taking the variation of L, you find that the condition $\delta_\alpha L = 0$ is equivalent to

$$\delta_\alpha \int \mathcal{L} d\tau = 0, \tag{7}$$

where $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. For the Schwarzshild metric, we have

$$\mathcal{L} = -A(r)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{A(r)}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2. \tag{8}$$

We will show explicitly the equation of motion for t.

$$\delta_t \int \mathcal{L} d\tau = -2\int A(r)\frac{dt}{d\tau}\delta_t\left[\frac{dt}{d\tau}\right] d\tau = -2\int A(r)\frac{dt}{d\tau}\frac{d\delta_t t}{d\tau} d\tau = -2\int \frac{d}{d\tau}\left[A(r)\frac{dt}{d\tau}\right] d\tau \delta t, \tag{9}$$

where in the last step we used integration by part (we let $u = A(r)\frac{dt}{d\tau}$ and $dv = \frac{d\delta_t t}{d\tau}d\tau$, Then, $du = \frac{d}{d\tau}\left[A(r)\frac{dt}{d\tau}\right] d\tau$ and $v = \delta_t t$). Since (9) vanishes for a geodesics, we find for the coordinate t

$$\frac{d}{d\tau}\left[A(r)\frac{dt}{d\tau}\right] = 0. \tag{10}$$

One can proceed in an analogue way for the coordinate θ and φ. The results is given by (3) and (4).

Proof of $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 = 0$:

Variation with Respect to θ results in:

$$\begin{aligned}
\delta_\theta \int \mathcal{L} d\tau &= \int \delta_\theta \left[r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right] d\tau \\
&= \int \delta_\theta \left[r^2\left(\frac{d\theta}{d\tau}\right)^2\right] + \int \delta_\theta\left[r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right] d\tau \\
&= \int 2r^2\frac{d\theta}{d\tau}\frac{d}{d\tau}(\delta\theta) d\tau + \int 2r^2\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2 \delta\theta d\tau \\
&= -\int 2\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right)\delta\theta d\tau + \int 2r^2\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2\delta\theta d\tau \quad \text{, (integration by parts)} \\
&= -2\int \left[\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]\delta\theta d\tau
\end{aligned}$$

and thus the condition $\delta_\theta \int \mathcal{L} d\tau = 0$ leads to the equation (3). The derivation of (4) is the same as (2).

Q2. Conclude that, without loss of generality, we can consider that the motion occurs in the plane $\theta = \pi/2$, and that there are two constants of motion $E \equiv A(r)\dot{t}$, $L \equiv r^2\dot{\varphi}$. Do you understand the fundamental origin of those constants of motion?

Solution Q2

We can always choose a plane containing $r = 0$ and the initial velocity of the particle. We can set the initial condition such that $\theta_{\text{in}} = \pi/2$. Then, from Eq. (3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) = 0, \tag{11}$$

which admits $d\theta/d\tau = 0$ as solution. This means that a trajectory that is initially in the equatorial plane will remain in the equatorial plane. For $\theta = \pi/2$, the equation of motion for the particle is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[A(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = 0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right) = 0, \tag{13}$$

i.e. the quantities $E = A(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $L = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$ are conserved. These constant of motion originate from the symmetry of the metric: E is the Noether [1] charge associated to the staticity of the system, while L is the charge associated with the spherical symmetry. They can be though of as the energy and angular momentum of the particle.

Q3. Using the normalisation of the four-velocity of the particle and the constants of motion, derive a first-order equation of motion for $r(\tau)$. Introducing the Binet variable $u \equiv 1/r$, show that the trajectory $u(\varphi)$ satisfies the modified second Binet equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2. \tag{14}$$

In which regime do we recover the Newtonian case?

Solution Q3

We fix $\theta = \pi/2$. Using the normalisation for the 4-velocity $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$, we have

$$\frac{1}{A(r)}\dot{r}^2 = -1 + A(r)\dot{t}^2 - r^2\dot{\varphi}^2. \tag{15}$$

Multiplying (15) by $A(r)$, we can write $\dot{t}$ and $\dot{\varphi}$ in terms of E and L, respectively. Then, we obtain the following equation of motion for r:

$$\dot{r}^2 + A(r)\left(1 + \frac{L^2}{r^2}\right) = E^2 \quad , \quad (r^2\dot{\varphi}^2 = \tfrac{r^2\dot{\varphi}.r^2\dot{\varphi}}{r^2} = \tfrac{L^2}{r^2})$$

Introduce $u = 1/r$, so that $\dot{u} = -r^{-2}\dot{r}$, we have

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{\dot{u}}{\dot{\varphi}} = -\frac{\dot{r}}{r^2\dot{\varphi}} = -\frac{\dot{r}}{L}$$

If we substitute $\dot{r}^2 = L^2\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2$ in the previous equation, we obtain

$$L^2\left(\frac{du}{d\varphi}\right)^2 + A(u)(1 + u^2L^2) = E^2, \tag{16}$$

[1] Noether's theorem links symmetries in physical systems to conservation laws.

where $A(u) = 1 - r_s u = 1 - 2GMu$. If we take the derivative with respect to φ, the final result gives

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2, \tag{17}$$

which is the relativistic Binet equation.

Proof of : $\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2$

Taking the derivative of (16) with respect to φ yields

$$\begin{aligned}
\frac{d}{d\varphi}\left[L^2\left(\frac{du}{d\varphi}\right)^2\right] + \frac{d}{d\varphi}[A(u)(1+u^2L^2)] &= 0\\
2L^2\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} + \frac{dA}{du}\frac{du}{d\varphi}(1+u^2L^2) + A(u)\frac{d}{d\varphi}(1+u^2L^2) &= 0\\
2L^2\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} - 2GM\frac{du}{d\varphi}(1+u^2L^2) + (1-2GMu)2uL^2\frac{du}{d\varphi} &= 0 \quad , \quad (A(u) = 1-2GMu)\\
\frac{du}{d\varphi}\left[2L^2\frac{d^2u}{d\varphi^2} - 2GM(1+u^2L^2) + (1-2GMu)2uL^2\right] &= 0
\end{aligned}$$

which means that the factor inside the bracket needs to vanish. We can rearrange and simplify the equation to isolate $\frac{d^2u}{d\varphi^2}$

$$\frac{d^2u}{d\varphi^2} = \frac{GM(1+u^2L^2) - uL^2(1-2GMu)}{L^2} = \frac{GM}{L^2} - u + 3GMu^2$$

In Newtonian gravity, the Binet equation is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{GM}{L^2}. \tag{18}$$

The relativistic version thus differs by the addition of the quadratic term $3GMu^2$. The Newtonian limit is reached for $3GMu << 1$.

We are going to solve eq. (14) using a perturbative technique called *multiple-scale expansion*, which is particularly useful to deal with non-linear equations. The underlying motivation is that the two terms of eq. (14) produce changes of u on distinct scales. This is clearer if we introduce the dimensionless quantity $U \equiv u/(GM/L^2) \sim 1$, which yields

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\varphi^2} + U = 1 + \varepsilon U^2, \qquad \varepsilon \equiv 3\left(\frac{GM}{L}\right)^2 \tag{19}$$

where we see that while the first term on the right-hand side produces changes of U over angular scales of order unity, the second term produces significant changes on angular scales of order $\varepsilon^{-1} \gg 1$. The multi-scale expansion consists in dealing with U as a function of two independent variables, $U(\varphi_0, \varphi_1)$, where $\varphi_0 = \varphi$ represents the 'fast' evolution and $\varphi_1 = \varepsilon\varphi$ the 'slow' evolution. We also consider a perturbative expansion of U itself, such that

$$U(\varphi) = U_0(\varphi_0, \varphi_1) + \varepsilon U_1(\varphi_0, \varphi_1) + \dots \tag{20}$$

Q4. By consistently expanding eq. (19) at first order in ε, show that

$$\frac{\partial^2 U_0}{\partial {\varphi_0}^2} + U_0 = 1 \tag{21}$$

$$\frac{\partial^2 U_1}{\partial {\varphi_0}^2} + U_1 = U_0^2 - 2\frac{\partial^2 U_0}{\partial \varphi_0 \partial \varphi_1}, \tag{22}$$

and give the general solution of eq. (21).

Solution Q4

Cumulative Effect Over Large Angular Scales

In order to see the cumulative effect of the nonlinear term εU^2, we need to consider changes in U over a large range of φ. If we look at changes in φ of the order of ε^{-1}, the small effects of εU^2 can ***accumulate*** to produce a significant change in U. Mathematically, if we consider a change in φ by an amount $\Delta\varphi \approx \varepsilon^{-1}$, the small term εU^2 gets effectively multiplied by $\Delta\varphi$, making its cumulative impact over this range comparable to the other terms in the dimensionless Binet equation.

Mathematical Interpretation

Considering a Taylor expansion of U around some ϕ_0 over a range $\Delta\phi$:

$$U(\phi) \approx U(\phi_0) + \left.\frac{dU}{d\phi}\right|_{\phi_0} \Delta\phi + \frac{1}{2}\left.\frac{d^2U}{d\phi^2}\right|_{\phi_0} (\Delta\phi)^2 + \dots$$

for small $\Delta\phi$, the terms involving higher powers of $\Delta\phi$ are negligible. However, when $\Delta\phi$ is of the order of ε^{-1}, these terms become significant, especially the term εU^2 in our equation.

We start by expanding the derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} = \left(\frac{\partial}{\partial\varphi_0} + \varepsilon\frac{\partial}{\partial\varphi_1}\right)^2 = \frac{\partial^2}{\partial{\varphi_0}^2} + 2\varepsilon\,\frac{\partial^2}{\partial\varphi_0\partial\varphi_1} + \dots$$

when substituting the expanded form of the derivative and $U = U_0 + \varepsilon U_1$ into equation (19), and also neglecting terms of order ϵ^2 and higher, we get

$$\left[\frac{\partial^2 U_0}{\partial {\varphi_0}^2} + U_0 - 1\right] + \varepsilon\left[\frac{\partial^2 U_1}{\partial {\varphi_0}^2} + U_1 + 2\frac{\partial^2 U_0}{\partial\varphi_0\partial\varphi_1} - U_0^2\right] = 0,$$

which gives the desired result.
The solution of the zeroth order is simply $U_0 = 1 + A(\varphi_1)\cos\varphi_0 + B(\varphi_1)\sin\varphi_0$.

We recognize in eq. (22) the equation for a forced harmonic oscillator, with resonance frequency $\omega_0 = 1$. For the perturbative expansion $U_0 + \varepsilon U_1$ to be meaningful, we have to prevent resonances in U_1, by ensuring that the forcing terms with frequency ω_0 on the right-hand side of eq. (22) vanish. Such terms are called *secular*.

Q5. Inserting your general solution for U_0 in eq. (22), identify the secular terms. Show that the non-resonance condition implies $A - B' = A' + B = 0$, where $A(\varphi_1)$ and $B(\varphi_1)$ refer to the unknown functions remaining in the general solution for U_0.

Solution Q5

Preventing Resonances in Perturbative Expansion

Resonance in a harmonic oscillator occurs when the frequency of the forcing term matches the natural frequency of the system, leading to a large increase in the amplitude of oscillations. For the perturbative expansion to be valid and meaningful, it is crucial to prevent resonances from occurring in U_1. If there are terms on the right-hand side of the equation that oscillate at the resonance frequency ω_0, they could cause U_1 to grow without bound, invalidating the perturbative approach. This means ensuring that any terms in the forcing part of the equation ($U_0^2 - 2\frac{\partial^2 U_0}{\partial\varphi_0\partial\varphi_1}$) that oscillate with the frequency $\omega_0 = 1$ must be eliminated or made to vanish. These terms are referred to as secular terms.

By calculating $U_0^2 - 2\frac{\partial^2 U_0}{\partial\varphi_0\partial\varphi_1}$ based on our solution $U_0 = 1 + A(\varphi_1)\cos\varphi_0 + B(\varphi_1)\sin\varphi_0$, for the first term we get

$$U_0^2 = 1 + A^2\cos^2\varphi_0 + B^2\sin^2\varphi_0 + 2A\cos\varphi_0 + 2B\sin\varphi_0 + 2AB\sin\varphi_0\cos\varphi_0$$

and for the second term

$$\begin{aligned}
-2\frac{\partial^2 U_0}{\partial\varphi_0\partial\varphi_1} &= -2\frac{\partial}{\partial\varphi_1}\left(\frac{\partial U_0}{\partial\varphi_0}\right) \\
&= -2\frac{\partial}{\partial\varphi_1}\left(-A(\varphi_1)\sin\varphi_0 + B(\varphi_1)\cos\varphi_0\right) \\
&= -A'(\varphi_1)\sin\varphi_0 + B'(\varphi_1)\cos\varphi_0
\end{aligned}$$

The secular terms, i.e. the terms oscillating with φ_0 with unity angular frequency, in $U_0^2 - 2\partial^2 U_0/\partial\varphi_0\partial\varphi_1$, are then identified

$$2A(\varphi_1)\cos\varphi_0 + 2B(\varphi_1)\sin\varphi_0 - 2\left[-A'(\varphi_1)\sin\varphi_0 + B'(\varphi_1)\cos\varphi_0\right].$$

Having them vanishing then implies

$$(A - B')\cos\varphi_0 + (A' + B)\sin\varphi_0 = 0,$$

which must be true for any φ_0. The family $(\cos, \sin)$ being linearly independent, this implies that both pre-factors $A - B'$, $A' + B$ must vanish.

Q6. Solve for A, B, and conclude that the lowest order solution for the trajectory of the particle can be written

$$r = \frac{L^2}{GM}\frac{1}{1 + e\cos\left(1 - \frac{3G^2M^2}{L^2}\right)\varphi} \tag{23}$$

Solution Q6

Taking the derivative of $A - B' = A' + B = 0$, we find that both functions satisfy $y'' + y = 0$. Reinjecting the general solution into $A - B' = A' + B = 0$ then yields

$$\begin{aligned}
A(\varphi_1) &= \alpha\cos\varphi_1 + \beta\sin\varphi_1, \\
B(\varphi_1) &= -\beta\cos\varphi_1 + \alpha\sin\varphi_1,
\end{aligned}$$

where α, β are arbitrary constants. Inserting these into the expression of $U_0 = 1 + A(\varphi_1)\cos\varphi_0 + B(\varphi_1)\sin\varphi_0$, we find

$$\begin{aligned}
U_0(\varphi_0, \varphi_1) &= 1 + \alpha\left(\cos\varphi_1\cos\varphi_0 + \sin\varphi_1\sin\varphi_0\right) + \beta(\sin\varphi_1\cos\varphi_0 - \cos\varphi_1\sin\varphi_0) \\
&= 1 + \alpha\cos(\varphi_0 - \varphi_1) + \beta\sin(\varphi_0 - \varphi_1)
\end{aligned}$$

or assuming that

$$e\cos(\psi) = \alpha \quad , \quad e\sin(\psi) = \beta$$

we can write

$$U_0(\varphi_0, \varphi_1) = 1 + e\,\cos(\varphi_0 - \varphi_1 - \psi).$$

Replacing φ_0, φ_1 by their definitions, we end up with the lowest order solution

$$u(\varphi) = \frac{GM}{L^2}\left[1 + e\cos\left(1 - \frac{3G^2M^2}{L^2}\right)\varphi\right] \tag{24}$$

where we have set $\psi = 0$ without loss of generality.

Q7. Interpret this solution geometrically. Show that at each of its revolution around the Sun, Mercury's perihelion is shifted by an angle

$$\delta = \frac{6\pi GM_\odot}{a(1-e^2)}, \tag{25}$$

where a is the semi-major axis of the Keplerian trajectory. Calculate the shift per century, using that $GM_\odot = 1.5$ km, and the orbital characteristics of Mercury (☿): semi-major axis $a_☿ = 5.8 \times 10^7$ km, eccentricity $e_☿ = 0.2$, and orbital period $T_☿ = 88$ days.

Solution Q7

The solution (23), which in terms of $r(\varphi)$ reads

$$r(\varphi) = \frac{L^2/GM}{\left[1 + e\cos\left[\left(1 - \frac{3G^2M^2}{L^2}\right)\varphi\right]\right]}, \tag{26}$$

can be understood as an ellipse on a system of axis which rotates in the same direction as the planet's motion, with inclination $\delta = 3G^2M^2\varphi/L^2$ with respect to the initial set of axes. After one revolution, this shift thus reads

$$\delta = \frac{6\pi G^2M^2}{L^2}.$$

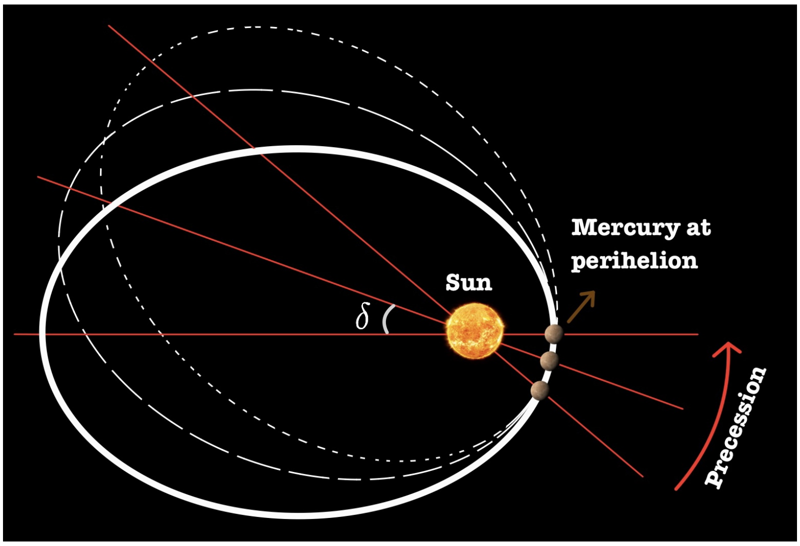

Figure 1 Schematic of Mercury's precession. A better representation can be found here.

We express L as a function of the Keplerian trajectory. The maximum and minimum values of r are given by

$$r_\pm = \frac{L^2}{GM}\frac{1}{1 \pm e}.$$

Their sum give the length or the major axis

$$2a = r_+ + r_- = \frac{L^2}{GM}\frac{2}{1-e^2} \qquad \text{whence} \qquad \delta = \frac{6\pi G^2 M^2}{L^2} = \frac{6\pi GM}{a(1-e^2)}.$$

When applied to the trajectory of Mercury around the Sun, we conclude that the angular velocity of the perihelion of the Keplerian trajectory is

$$\begin{aligned}\Omega_{☿} &\equiv \delta_{☿} \;\text{[in units of Mercury's orbits]} \\ &= \delta_{☿} \times 100.0 \times \frac{365}{T_{☿}\;\text{[days]}} \;\text{[per century]} \\ &= \frac{6\pi GM_{\odot}}{a_{☿}(1-e_{☿}^2)} \times 100.0 \times \frac{365}{88} \;\text{[per century]} = 43\,\text{arcsec/century}\end{aligned}$$

which is exactly the observed residual precession.