Gravitational Lensing

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1 The lens equation

Consider a spherically symmetric distribution of mass M generating a weak gravitational field. By solving the null geodesic equation in the corresponding spacetime, you have seen that a light ray emitted and observed very far away from the mass is deflected by an angle

$$\alpha = \frac{4GM}{b}, \tag{1}$$

where b is the impact parameter, which corresponds at lowest order to the minimal distance between the photon and the mass in the trajectory (see fig. 1).

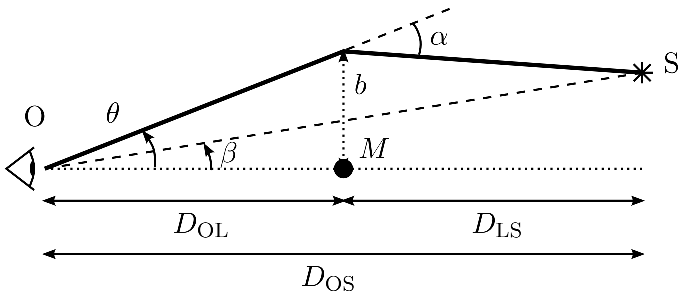


Figure 1 Deflection of light by a spherically symmetric mass M. O, S respectively denote the observer and the source. The angles β, θ are the unlensed and lensed position of the image of S in the observer's celestial sphere, while α is the deflection angle given by eq. (1). Finally, $D_{\mathrm{OL}}, D_{\mathrm{LS}}$ and D_{OS} denote, respectively, the observer-lens, lens-source, and observer-source distances.

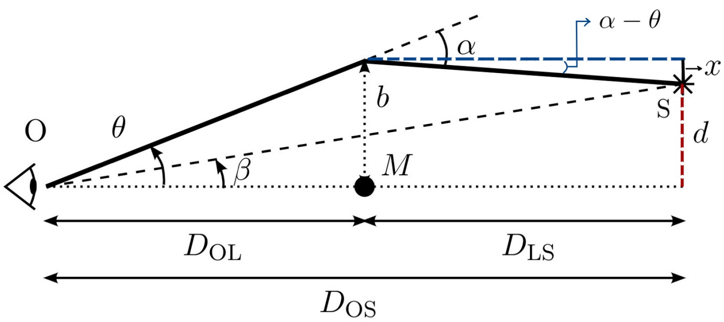
Q1. Assuming that $\beta, \theta \ll 1$, show that

$$\beta = \theta - \frac{\theta_{\mathrm{E}}^2}{\theta}. \tag{2}$$

where you will express θ_{E}, called the *Einstein radius*, as a function of $G, M, D_{\mathrm{OL}}, D_{\mathrm{LS}}, D_{\mathrm{OS}}$. Equation (2) is known as the *lens equation*.

Solution Q1

Let us call d the distance between the source S and the axis (OM).



On one hand for small angles, we have

$$d = \beta D_{\rm OS}$$

On the other hand we can write

$$d = b - x = \theta D_{\rm OL} - (\alpha - \theta) D_{\rm LS}$$

Substituting for α and dividing by $D_{\rm OS}$ we find

$$\begin{aligned} \beta &= \theta \frac{D_{\rm OL}}{D_{\rm OS}} + (\theta - \alpha)\frac{D_{\rm LS}}{D_{\rm OS}} \\ &= \theta\left(\frac{D_{\rm OL}}{D_{\rm OS}} + \frac{D_{\rm LS}}{D_{\rm OS}}\right) - \frac{4GMD_{\rm LS}}{bD_{\rm OS}} \\ &= \theta - \frac{4GMD_{\rm LS}}{D_{\rm OL}D_{\rm OS}}\frac{1}{\theta} \quad , \quad (b = \theta D_{\rm OL}) \end{aligned}$$

which is the lens equation if we define $\theta_{\rm E}^2 \equiv 4GMD_{\rm LS}/(D_{\rm OL}D_{\rm OS})$.

Q2. Solve eq. (2) for θ. How many solutions are there? How do you understand this? Why do not we see multiple images of every light source in our daily life?

Solution Q2

Multiplying eq. (2) by θ and solving this 2nd-degree polynomial equation, we find that there are always two solutions

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\rm E}^2}\right),$$

which corresponds to the fact that light can pass 'above' or 'below' the lens in fig. 1. Note that there is always one solution for which $|\theta_\pm| \leq \theta_{\rm E}$ and the other with $|\theta_\pm| \geq \theta_{\rm E}$. If the lens is not a point but an extended opaque object with angular size Θ, both solutions can be observed only if $\Theta < \theta_{\rm E}$. If not, one of the possible rays will be blocked by the lens itself. When the lens is smaller than its Einstein radius, $\Theta < \theta_{\rm E}$, it is called supercritical, and can produce multiple images.

Such a situation never happens in everyday life. The maximum Einstein radius of an object of mass $M \sim$ kg located at a distance $D \sim$ m from us is indeed

$$\theta_{\rm E} \approx 3 \times 10^{-12} \sqrt{\frac{M}{1\,\text{kg}} \frac{1\,\text{m}}{D}}\ \text{deg},$$

which would require the object to be really tiny. However, for a star ($M \sim M_\odot$) in our galaxy $D \sim 10\,\text{kpc}$, we have

$$\theta_{\rm E} \approx 0.9 \sqrt{\frac{M}{M_\odot}\frac{10\,\text{kpc}}{D}}\ \text{mas},$$

which can be larger than the apparent size of a star. The same thing happens for galaxies on extragalactic distances.

Q3. The image in fig. 2 has been observed by the Hubble space telescope. Explain this picture. What is the radius of the ring? How can this be used in astrophysics?

Solution Q3

When the source is exactly aligned with the lens ($\beta = 0$), then $\theta = \pm\theta_{\rm E}$. However, in this case, the situation is symmetric under rotation about the optical axis. In other words, there are not just two solutions, but an infinity, around the lens. What the observer sees is not just two images, but a circle of images with apparent radius $\theta_{\rm E}$ around the lens. This is what is shown in fig. 2.

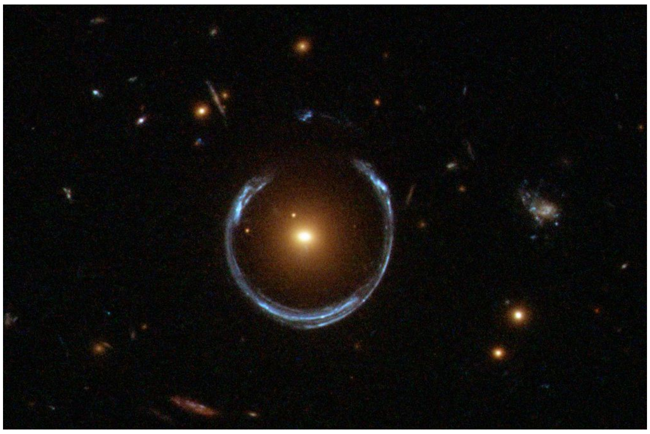


Figure 2 A spectacular gravitational-lensing observation by the Hubble space telescope.

2 Gravitational amplification

Consider a small but non-punctual light source. In this exercise, we are going to show that its apparent luminosity can be amplified by gravitational lensing. A first step consists in slightly generalising the lens equation established in the previous exercise. In the approximation of the small angles, we can consider a fictitious plane in which we represent the angular positions of the sources and images by 2-dimensional vectors. In such conditions, the lens equation is straightforwardly generalised as

$$\boldsymbol{\beta} = \left(1 - \frac{\theta_{\rm E}^2}{\theta^2}\right)\boldsymbol{\theta}, \tag{3}$$

where $\theta \equiv |\boldsymbol{\theta}|$.

The *amplification matrix* $\boldsymbol{\mathcal{A}}$ is defined as the Jacobian matrix of the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\beta}$, that is

$$\mathcal{A}_{ab} \equiv \frac{\partial \beta_a}{\partial \theta_b}. \tag{4}$$

The altitude of indices a, b does not matter, as $\boldsymbol{\theta}, \boldsymbol{\beta}$ live in a two-dimensional Euclidean space.

Q1. Calculate $\mathcal{A}_{ab}$ explicitly, and show that its determinant reads

$$\det \boldsymbol{\mathcal{A}} = 1 - \left(\frac{\theta_{\rm E}}{\theta}\right)^4. \tag{5}$$

Solution Q1

Note that we have $\theta^2 = \theta_1^2 + \theta_2^2$. For $a = b = 1$ we can write

$$\mathcal{A}_{11} = \frac{\partial \beta_1}{\partial \theta_1} = \frac{\partial\left[\left(1 - \frac{\theta_E^2}{\theta_1^2+\theta_2^2}\right)\theta_1\right]}{\partial \theta_1} = \frac{\partial\left(\theta_1 - \theta_1 \frac{\theta_E^2}{\theta_1^2+\theta_2^2}\right)}{\partial \theta_1} = 1 - \frac{\theta_E^2}{\theta^2} + 2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1^2$$

where in the last step we have used the relation

$$y = \frac{\theta_{\rm E}^2}{\theta_1^2 + \theta_2^2} = \frac{u}{v} \longrightarrow \frac{dy}{d\theta_1} = \frac{u'v - uv'}{v^2} = -2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1$$

Similarly for other components we find

$$\mathcal{A}_{12} = \mathcal{A}_{21} = 2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1\theta_2 \quad , \quad \mathcal{A}_{22} = 1 - \frac{\theta_{\rm E}^2}{\theta^2} + 2\frac{\theta_{\rm E}^2}{\theta^4}\theta_2^2$$

So, in a general form we can write

$$\mathcal{A}_{ab} = +\frac{2\theta_{\rm E}^2}{\theta^4}\,\theta_a\theta_b + \left(1 - \frac{\theta_{\rm E}^2}{\theta^2}\right)\delta_{ab},$$

or, under a matrix form,

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\theta}) = \begin{bmatrix} 1 - \frac{\theta_{\rm E}^2}{\theta^2} + 2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1^2 & +2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1\theta_2 \\ +2\frac{\theta_{\rm E}^2}{\theta^4}\theta_1\theta_2 & 1 - \frac{\theta_{\rm E}^2}{\theta^2} + 2\frac{\theta_{\rm E}^2}{\theta^4}\theta_2^2 \end{bmatrix}$$

The straightforward calculation of its determinant yields

$$\det \boldsymbol{\mathcal{A}} = 1 - \left(\frac{\theta_{\rm E}}{\theta}\right)^4.$$

Q2. Justify, from the geometrical meaning of the determinant of a matrix, that if a small (but non-punctual) image is observed in $\boldsymbol{\theta}$, then

$$\mu \equiv \frac{\Omega}{\Omega_0} = \left|\frac{1}{\det \boldsymbol{\mathcal{A}}}\right|, \tag{6}$$

where Ω is the apparent size of the lensed image, and Ω_0 would be its size if there were no lensing. The quantity μ is called the magnification.

Solution Q2

Lensing can be considered as a mapping $\boldsymbol{\beta} \to \boldsymbol{\theta} = L(\boldsymbol{\beta})$ from the source space to the image space. By definition, an infinitesimal volume $\mathrm{d}^2\boldsymbol{\beta}$ in the source space is then mapped to an infinitesimal volume $\mathrm{d}^2\boldsymbol{\theta}$ in the image space, such that

$$\mathrm{d}^2\boldsymbol{\theta} = |J|\,\mathrm{d}^2\boldsymbol{\beta},$$

where $J = \det[\partial\boldsymbol{\theta}/\partial\boldsymbol{\beta}]$ is the Jacobian of L. Comparing this to the definition of the amplification matrix, we conclude that $J = 1/\det \boldsymbol{\mathcal{A}}$. Now $\mathrm{d}^2\boldsymbol{\beta} = \Omega_0$ has to be interpreted as the apparent size of the source without lensing, while $\mathrm{d}^2\boldsymbol{\theta}$ is the apparent size of the image, which yields $\Omega = \Omega_0/|\det \boldsymbol{\mathcal{A}}|$ as required.

It can be shown, although non-trivially, that μ is also the ratio between the observed luminous intensity I and its unlensed counterpart I_0.

Q3. Why is *amplification matrix* a very confusing name for $\boldsymbol{\mathcal{A}}$?

Solution Q3

Since $\mu = I/I_0 = 1/|\det \boldsymbol{\mathcal{A}}|$, the larger $|\det \boldsymbol{\mathcal{A}}|$ the fainter the image. This is rather contrary to what we would expect from a quantity called ‘amplification”, i.e. that a large determinant would imply a large amplification of light, not a demagnification.

Q4. Show that the magnifications of the two images of a small extended source read

$$\mu_\pm = \left|\frac{1}{2} \pm \frac{u^2+2}{2u\sqrt{u^2+4}}\right|, \tag{7}$$

where $u \equiv \beta/\theta_{\rm E}$. What is the total magnification, taking into account both images?

Solution Q4

We start from the expression of the magnification as a function of the image position, $\mu_\pm = 1/\left|1-\theta_{\rm E}^4/\theta_\pm^4\right|$, and we substitute the formula for $\theta_\pm$,

$$\mu_\pm = \left|\frac{1}{1-\left(\frac{2\theta_{\rm E}}{\beta\pm\sqrt{\beta^2+4\theta_{\rm E}^2}}\right)^4}\right| = \left|\frac{1}{1-\left(\frac{2}{u\pm\sqrt{u^2+4}}\right)^4}\right| = \left|\frac{\left(u\pm\sqrt{u^2+4}\right)^4}{\left(u\pm\sqrt{u^2+4}\right)^4-16}\right|$$

where we introduced $u\equiv\beta/\theta_{\rm E}$. We first note that

$$\left(u\pm\sqrt{u^2+4}\right)^2 = 2\left(u^2+2\pm u\sqrt{u^2+4}\right);$$

then we manipulate the denominator

$$\begin{aligned}\left(u\pm\sqrt{u^2+4}\right)^4-16 &= \left[\left(u\pm\sqrt{u^2+4}\right)^2-4\right]\times\left[\left(u\pm\sqrt{u^2+4}\right)^2+4\right]\\ &= 2\left(u^2+2\pm u\sqrt{u^2+4}-2\right)\times 2\left(u^2+2\pm u\sqrt{u^2+4}+2\right)\\ &= 4\left(u^2\pm u\sqrt{u^2+4}\right)\left(u^2+4\pm u\sqrt{u^2+4}\right)\\ &= 4u\left(u\pm\sqrt{u^2+4}\right)\sqrt{u^2+4}\left(\sqrt{u^2+4}\pm u\right)\\ &= \pm 4u\sqrt{u^2+4}\left(u\pm\sqrt{u^2+4}\right)^2,\end{aligned}$$

which finally gives

$$\mu_\pm = \left|\frac{\left(u\pm\sqrt{u^2+4}\right)^2}{\pm 4u\sqrt{u^2+4}}\right| = \left|\frac{u^2+2\pm u\sqrt{u^2+4}}{\pm 2u\sqrt{u^2+4}}\right| = \left|\frac{1}{2}\pm\frac{u^2+2}{2u\sqrt{u^2+4}}\right|.$$

Since $|u^2+2| > \left|u\sqrt{u^2+4}\right|$ (check by taking the square of it) we conclude that the absolute value brings a minus sign to one of the $\mu_\pm$. Let us assume, to alleviate notation, that $u>0$, then the total magnification reads

$$\mu_{\rm tot} = \mu_+ + \mu_- = \frac{u^2+2}{u\sqrt{u^2+4}} > 1,$$

so that the set of both images is always magnified.

As an illustration of this phenomenon, the graph of fig. 3 has been obtained by monitoring the luminosity of a star (S) for about three years. Between, roughly, day 400 and day 600, another star (L) passed on the line of sight, producing an enhancement of the apparent luminosity of S.

Q5. Recalling that the magnitude of a star is given by $m=-2.5\log_{10} I + \mathrm{cst}$, where I is the observed luminous intensity, determine the minimum value of u in this event.

Solution Q5

Let us call $I_{\rm peak}, m_{\rm peak}$, respectively, intensity and magnitude of the image at the peak of this *microlensing event*. The difference in magnitude with the unlensed situation is then

$$\Delta m = m_0 - m_{\rm peak} = 2.5\log_{10}\left(\frac{I_{\rm peak}}{I_0}\right). \tag{8}$$

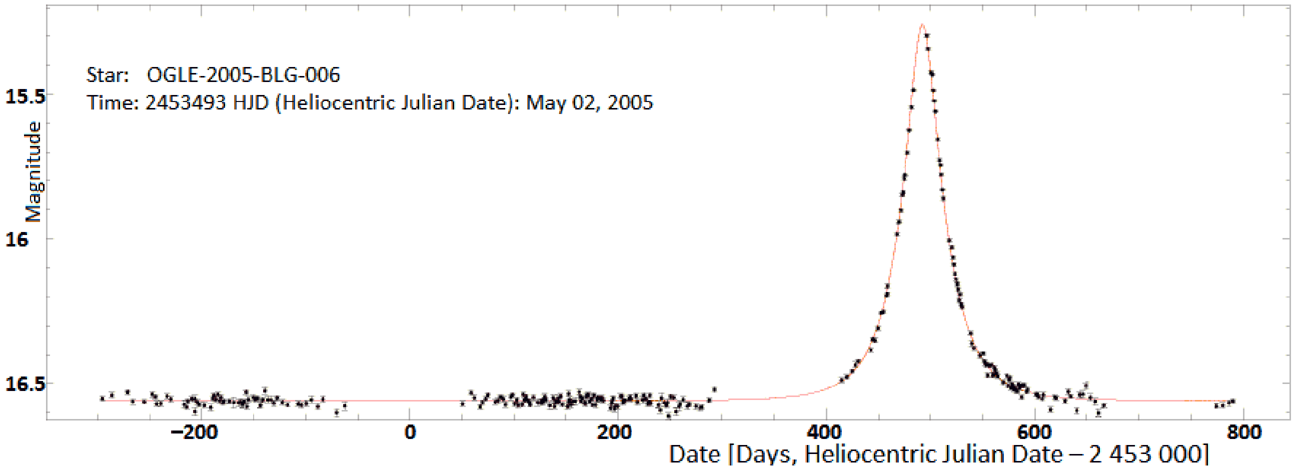


Figure 3 Gravitational amplification of a star by another star acting as a lens.

Hence the peak magnification reads

$$\mu_{\text{peak}} = \frac{I_{\text{peak}}}{I_0} = 10^{\frac{2\Delta m}{5}} \approx 3.3 \tag{9}$$

as we read $\Delta m \approx 1.3$ from fig. 3. Besides, we can invert the equation giving the total magnification as a function of u,

$$u = \sqrt{\frac{2\mu_{\text{peak}}}{\sqrt{\mu_{\text{peak}}^2 - 1}} - 2}$$

so that $u_{\text{peak}} \approx 0.3$ here. The source has really entered the Einstein radius of the lens.