Gravitational Waves

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This exercise is dedicated to gravitational waves, their effect on matter, and their detection. This subject is quite topical: the first detection by LIGO occurred on the 14th of September 2015, followed by six other events from 2015 to 2017. For their decisive contribution to this breakthrough, R. Weiss, K. Thorne, and B. Barish shared the 2017 Nobel Prize in Physics.

1 Tidal field of a gravitational wave

Let us consider a plane monochromatic gravitational wave propagating in the direction $\boldsymbol{\partial}_z$. You have seen in the lectures that the corresponding space-time metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with, in the transverse-traceless gauge (TT),

$$h^{\rm TT}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}\mathrm{e}^{\mathrm{i}k_\sigma x^\sigma} + \text{c.c.}, \qquad [k_\mu] = [-\omega, 0, 0, \omega] \qquad [\varepsilon_{\mu\nu}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

where c.c. indicates the complex conjugate of the expression, ω is the frequency of the wave, and $h_+, h_\times$ are constant numbers indicating its strain amplitude and polarisation.

Q1. Express the Riemann tensor of this space-time, at linear order, as a function of $h_{\mu\nu,\rho\sigma}$.

Solution Q1

At linear order in $h_{\mu\nu}$, you have seen in the lecture notes that the Christoffel symbols read (note that we have $\partial_\lambda \eta_{\mu\nu} = 0$ and we use $\eta_{\mu\nu}$ to raise and lower the indices not the full $g_{\mu\nu}$, see lecture notes)

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(h^\alpha_{\mu,\nu} + h^\alpha_{\nu,\mu} - h_{\mu\nu}^{;\alpha}\right) + \mathcal{O}(h^2).$$

or with all indices down (using $\eta_{\alpha\rho}$),

$$\eta_{\alpha\rho}\Gamma^\alpha_{\mu\nu} = \Gamma_{\rho\mu\nu} = \frac{1}{2}\left(h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}\right) + \mathcal{O}(h^2).$$

We then find, for the Riemann tensor,

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\mu\nu\rho,\sigma} + \mathcal{O}(h^2) \\ &= \frac{1}{2}\left(h_{\mu\nu,\sigma\rho} + h_{\mu\sigma,\nu\rho} - h_{\nu\sigma,\mu\rho}\right) - \frac{1}{2}\left(h_{\mu\nu,\rho\sigma} + h_{\mu\rho,\nu\sigma} - h_{\nu\rho,\mu\sigma}\right) \\ &= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right) + \mathcal{O}(h^2). \end{aligned}$$

Q2. How does $h_{\mu\nu}$ transforms under an infinitesimal coordinate transformation (gauge transformation)? Conclude that the Riemann tensor is gauge invariant.

Solution Q2

For an infinitesimal coordinate transformation $x^\mu \to y^\alpha \equiv x^\alpha - \xi^\alpha$, the metric reads

$$g_{\mu\nu}(x^\rho) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g'_{\alpha\beta}(y^\gamma)$$

where $g'_{\alpha\beta}$ is the metric expressed in the new coordinate system. Note that we use a prime to be able to distinguish it from $g_{\mu\nu}$ as we will need to use the same indices for both quantities in what follows. Using the fact that

$$\frac{\partial y^\alpha}{\partial x^\mu} = \delta^\alpha_\mu - \xi^\alpha{}_{,\mu}$$

plugging this into the metric transformation and keeping only the first-order term in ξ we find

$$\begin{aligned} g_{\mu\nu} &= (\delta^\alpha_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta_\nu - \xi^\beta{}_{,\nu}) g'_{\alpha\beta} \\ &= \delta^\alpha_\mu \delta^\beta_\nu g'_{\alpha\beta} - \delta^\alpha_\mu \xi^\beta{}_{,\nu} g'_{\alpha\beta} - \xi^\alpha{}_{,\mu}\delta^\beta_\nu g'_{\alpha\beta} + \mathcal{O}(\xi^2) \\ &= g'_{\mu\nu} - \xi^\beta{}_{,\nu} g'_{\mu\beta} - \xi^\alpha{}_{,\mu} g'_{\alpha\nu} \end{aligned}$$

at first order we thus find

$$\begin{aligned} h_{\mu\nu}(x^\rho) &= h'_{\mu\nu}(x^\rho) - \xi^\beta{}_{,\nu}\eta'_{\mu\beta} - \xi^\alpha{}_{,\mu}\eta'_{\alpha\nu} \\ &= h'_{\mu\nu}(x^\rho) - \xi_{\mu,\nu} - \xi_{\nu,\mu}, \end{aligned}$$

Note that ξ and $h_{\mu\nu}$ are of order ϵ, so we neglect their higher-order products like ξ^2 and $\xi h_{\mu\nu}$ which are of order ϵ^2. Therefore we have the transformation $h_{\mu\nu} \to h_{\mu\nu} + 2\xi_{(\mu,\nu)}$ of the metric perturbation under a gauge transformation. Inserting this in the expression of the Riemann tensor, we find that $R_{\mu\nu\rho\sigma} \to R_{\mu\nu\rho\sigma}$. In other words, the Riemann tensor is gauge-invariant.

In order to understand the effect of a gravitational wave on the motion of matter, it is instructive to use Fermi normal coordinates, which can be seen in the present context as a particular gauge. In the vicinity of any time-like geodesic, the metric perturbation reads

$$h^{\mathrm{F}}_{00} = -R_{0i0j}x^i x^j + \mathcal{O}(x^3) \tag{2}$$

$$h^{\mathrm{F}}_{0i} = -\frac{1}{3}(R_{0jik} + R_{0kij})x^j x^k + \mathcal{O}(x^3) \tag{3}$$

$$h^{\mathrm{F}}_{ij} = -\frac{1}{3}(R_{ikj\ell} + R_{i\ell jk})x^k x^\ell + \mathcal{O}(x^3). \tag{4}$$

Q3. Show that, in this gauge, the four-acceleration $\alpha^\mu \equiv u^\nu \nabla_\nu u^\mu$ of a non-relativistic particle reads, at lowest order in its velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$,

$$\alpha^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} - \frac{1}{3}(R_{0jik,0} + R_{0kij,0})x^k x^j + \frac{1}{2}(R_{0k0j}x^k x^j)_{,i}. \tag{5}$$

Solution Q3

The four-velocity of the particle can be written as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$$

Using the fact that $g_{\mu\nu}u^\mu u^\nu = -1$, at lowest order in v, h we can write

$$\begin{aligned}
g_{\mu\nu}u^\mu u^\nu &= g_{00}(u^0)^2 + 2g_{0i}u^0u^i + g_{ij}u^iu^j \\
&= (\eta_{00}+h_{00})\left(\frac{dt}{d\tau}\right)^2 + 2(\eta_{0i}+h_{0i})\left(\frac{dt}{d\tau}\right)^2 v^i + (\eta_{ij}+h_{ij})\left(\frac{dt}{d\tau}\right)^2 v^iv^j \\
&= (-1+h_{00})\left(\frac{dt}{d\tau}\right)^2 + \delta_{ij}v^iv^j\left(\frac{dt}{d\tau}\right)^2 \\
&= (-1+h_{00})\left(\frac{dt}{d\tau}\right)^2 + v^2\left(\frac{dt}{d\tau}\right)^2 \\
&\implies (-1+h_{00}+v^2)\left(\frac{dt}{d\tau}\right)^2 = -1 \quad \text{or} \quad \frac{dt}{d\tau} = (1-h_{00}-v^2)^{-\frac{1}{2}}
\end{aligned}$$

using binomial expansion we conclude that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1 + \frac{v^2}{2} + \frac{1}{2}h_{00}$$

which represents the energy per unit mass of the particle, with $v^2 = \delta_{ij}v^iv^j$. The four-acceleration then read

$$\begin{aligned}
\alpha^i &\equiv u^\nu\nabla_\nu u^i \\
&= u^\nu(\partial_\nu u^i + \Gamma^i_{\mu\nu}u^\mu) \\
&= u^\nu\partial_\nu u^i + \Gamma^i_{\mu\nu}u^\mu u^\nu \\
&= \frac{\mathrm{d}u^i}{\mathrm{d}\tau} + {\Gamma^i}_{\mu\nu}u^\mu u^\nu \quad , \quad (\text{where we used: } u^\nu = \frac{dx^\nu}{d\tau}) \\
&= \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}v^i\right) + \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\left({\Gamma^i}_{00} + 2v^j{\Gamma^i}_{j0} + v^jv^k{\Gamma^i}_{jk}\right) \\
&= \frac{\mathrm{d}v^i}{\mathrm{d}t} + {\Gamma^i}_{00} + \mathcal{O}(v^2, vh).
\end{aligned}$$

Finally, using Fermi normal coordinates, the only remaining Christoffel symbols read

$$\begin{aligned}
{}^{\mathrm{F}}{\Gamma^i}_{00} &= \frac{1}{2}\eta^{i\mu}\left(h^{\mathrm{F}}_{\mu0,0} + h^{\mathrm{F}}_{0\mu,0} - h^{\mathrm{F}}_{00,\mu}\right) \\
&= \frac{1}{2}\left(2h^{\mathrm{F}}_{i0,0} - h^{\mathrm{F}}_{00,i}\right) \\
&= -\frac{1}{3}(R_{0jik,0} + R_{0kij,0})x^kx^j + \frac{1}{2}(R_{0k0j}x^kx^j)_{,i},
\end{aligned}$$

which gives the desired result. Note that we changed the dummy index i to k in the expression for h^{F}_{00}.

Q4. Using the fact that the Riemann tensor is gauge invariant, and assuming that $|x| \ll \lambda$, where λ is the wavelength of the gravitational wave, show that we can write

$$\alpha^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \partial^i\Phi_{\mathrm{gw}}, \tag{6}$$

where you will express the pseudo-potential Φ_{gw} as a function of $h_+, h_\times, \omega$. *Hint*: Some terms in eq. (5) will be negligible.

Solution Q4

Let us compare the two classes of Riemann term that appear in the four-acceleration. Considering the expression for the Riemann tensor derived in **Q1**, we have

$$\frac{|(R_{0jik,0} + R_{0kij,0})x^kx^j|}{|(R_{0k0j}x^kx^j)_{,i}|} \sim \frac{|x|^2\partial^3h}{|x|^2\,\partial^3h + |x|\,\partial^2h} \sim \frac{|x|^2\partial^3h}{|x|\,\partial^2h} \sim \omega|x| \sim \frac{|x|}{\lambda} \ll 1.$$

we can thus neglect $(R_{0jik,0} + R_{0kij,0})x^k x^j$ compared to the second term, which can be rewritten as $\partial_i \Phi_{\rm gw}$, with

$$\begin{aligned}\Phi_{\rm gw} \equiv \frac{1}{2} R_{0i0j} x^i x^j &= \frac{1}{2} \cdot \frac{1}{2} \left(h_{0i,0j} + h_{0j,0i} - h_{00,ij} - h_{ij,00}\right) x^i x^j \\ &= \frac{\omega^2}{4} h_{ij} x^i x^j \quad \left({,}00 \text{ brings down twice the term } ik_0 \text{ in } h_{ij}^{\rm TT}\right) \\ &= \frac{\omega^2}{4} \left[h_{11} x^2 + h_{22} y^2 + h_{12} xy + h_{21} yx\right] \\ &= \frac{\omega^2}{8} \left[h_+ (y^2 - x^2) + 2 h_\times xy\right] \mathrm{e}^{\mathrm{i}\omega(z-t)} + \text{c.c.} \end{aligned} \tag{7}$$

Q5. How do you understand the effect of the gravitational wave on matter?

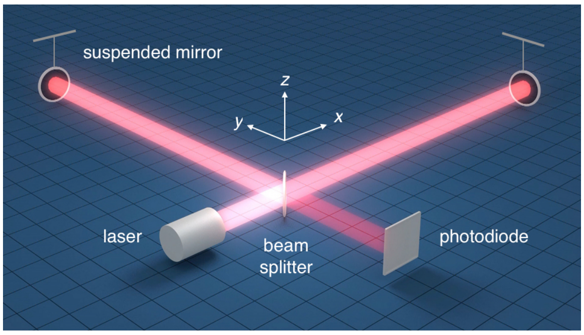
Solution Q5

It is always possible to turn the axes of the coordinate system in such a way that $h_\times = 0$, which makes the interpretation easier. As we can see, everything happens as if the particle were experiencing a gravitational potential which oscillates between a state where it is repulsive in the x direction and attractive in the y direction, and the opposite.

At a fixed time t, $\Phi_{\rm gw}$ is comparable to the tidal field created by, e.g., the Moon, on the Earth. As such, the effect of a gravitational wave is similar to having a Moon in a given direction, progressively disappears while appearing in the orthogonal direction, and so on.

2 A simplified interferometer

Let us now turn to how LIGO/Virgo actually detect gravitational waves, namely interferometers, as depicted on the right. A laser beam is split into two paths reflected by suspended mirrors. The reflected beams are then recombined on a screen where the interference pattern can be observed. In the following, we set the origin of the coordinate system at the beam splitter, and align the axes x and y with the two arms of the interferometer.



Q6. Give the expression of the luminous intensity I observed on the screen, as a function of the emitted intensity I_0, the lengths X, Y of the arms, and the wavelength of the laser.

Solution Q6

The luminous intensity is proportional to the averaged squared of the electromagnetic wave reaching the detector, which in our case is the superposition of two waves with the same amplitude $\sqrt{I_0/4}$:

$$E_1 = \sqrt{\frac{I_0}{4}} \cos(\omega t + \phi_1), \qquad E_2 = \sqrt{\frac{I_0}{4}} \cos(\omega t + \phi_2)$$

$$
\begin{aligned}
I \propto < E^2 > &= \left\langle \left[\sqrt{\frac{I_0}{4}} \cos(\omega t + \phi_1) + \sqrt{\frac{I_0}{4}} \cos(\omega t + \phi_2) \right]^2 \right\rangle \\
&= \left\langle \frac{I_0}{4} \cos^2(\omega t + \phi_1) + \frac{I_0}{4} \cos^2(\omega t + \phi_2) + \frac{I_0}{2} \cos(\omega t + \phi_1) \cos(\omega t + \phi_2) \right\rangle \\
&= \left\langle \frac{I_0}{4} \cos^2(\omega t + \phi_1) + \frac{I_0}{4} \cos^2(\omega t + \phi_2) + \frac{I_0}{4} \Big(\cos(\phi_1 - \phi_2) + \cos(2\omega t + \phi_1 + \phi_2) \Big) \right\rangle \\
&= \frac{I_0}{4} \Big(1 + \cos(\phi_1 - \phi_2) \Big)
\end{aligned}
$$

where we used trigonometric identity:

$$
\cos(\theta_1)\cos(\theta_2) = \frac{1}{2}\left[\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)\right],
$$

and the fact that:

$$
\langle \cos(\theta) \rangle = 0, \qquad \langle \cos^2(\theta) \rangle = 1/2,
$$

Finally, replacing the phase expressions

$$
\phi_1 = \frac{2\pi}{\lambda_l}.2X \quad , \quad \phi_2 = \frac{2\pi}{\lambda_l}.2Y \tag{8}
$$

The observed intensity is written as

$$
I = \frac{I_0}{4}\left(1 + \cos\phi\right).
$$

where $\phi = \frac{4\pi(X-Y)}{\lambda_l}$ is the phase difference.

Q7. Which default value Δ_0 of the optical path difference $\Delta \equiv 2(X-Y)$ makes the interferometer maximally sensitive to perturbations? In the following, we call $\delta \equiv \Delta - \Delta_0$.

Solution Q7

The interferometer is maximally sensitive to changes of the optical path difference when $|\mathrm{d}I/\mathrm{d}\Delta|$ is maximal. Since

$$
\left|\frac{\mathrm{d}I}{\mathrm{d}\Delta}\right| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}\Delta}\frac{\mathrm{d}I}{\mathrm{d}\phi}\right| = \left| -\frac{\pi I_0}{2\lambda_l}\sin\left(\frac{2\pi\Delta}{\lambda_l}\right)\right|,
$$

setting

$$
\frac{2\pi\Delta}{\lambda_l} = \frac{\pi}{2} + n\pi
$$

we conclude that the sensitivity is maximal for $\Delta_0 = \frac{\lambda_l}{2}(\frac{1}{2} + n)$, $n \in \mathbb{Z}$.

Let us now investigate the effect of a gravitational wave on this interferometer, and more precisely on the suspended mirrors. We have seen in § 1 that a gravitational wave can be seen as an oscillating gravitational pseudo-potential Φ_{gw}. This analysis was performed using Fermi normal coordinates, i.e. in a freely falling frame. Since current interferometers are ground-based (contrary to the future LISA), we first have to move to the Terrestrial frame.

Q8. Propose an approximate coordinate transformation from the coordinates x^μ of a non-relativistic particle in a freely falling frame, to its coordinates X^α in the terrestrial frame. Deduce the relationship between the four-accelerations of the particle in both frames, and conclude that its equation of motion reads

$$
\frac{\mathrm{d}V^i}{\mathrm{d}T} = -g^i - \partial^i\Phi_{\mathrm{gw}} + \frac{F^i}{m}, \tag{9}
$$

where $g^i = -g\delta^i_Z$ is the terrestrial gravitational acceleration, F^i is the sum of all non-gravitational forces applied to the particle, and m is mass.

Solution Q8

The coordinates of a **non-relativistic** particle in a freely-falling frame x^μ are related to its coordinates X^α in the terrestrial frame as

$$T = t, \quad X = x, \quad Y = y, \quad Z = z - \frac{gt^2}{2},$$

Z coordinate is adjusted to account for the effect of Earth's gravity. We assumed that the freely-falling frame is located at $z = 0$ and with zero velocity at $t = 0$. This assumption does not change the final result. We neglected special-relativistic effects, by assuming that $gt \ll 1$. Since the four-velocity is a vector, it transforms as $U^\alpha = (\partial X^\alpha/\partial x^\mu)u^\mu$, which yields $u^0 = U^0 = 1$,

$$V^i = v^i - g^i t, \qquad \text{and} \qquad \frac{\mathrm{d}v^i}{\mathrm{d}t} = \frac{\mathrm{d}V^i}{\mathrm{d}T} + g^i$$

as expected. Besides, the four-acceleration α^μ being also a four-vector, we have $A^\alpha = (\partial X^\alpha/\partial x^\mu)\alpha^\mu$, which gives $A^0 = \alpha^0$ and

$$A^i = \alpha^i - g^i t\alpha^0 \approx \alpha^i \ ,$$

since $gt \ll 1$ and $\alpha^0 = \mathrm{D}u^0/\mathrm{d}\tau \sim v\dot{v} \ll \alpha^i$ (in non-relativistic scenarios, the zeroth component of the four-acceleration (α^0) is small compared to its spatial components (α^i) because changes in temporal aspects of motion are minimal compared to changes in spatial motion at these lower speeds).

Proof of: $\alpha_0 \sim v\dot{v} \ll \alpha^i$

First, note that we have $u^\mu = \gamma(1, v^i)$, and γ is approximated as $\gamma \approx 1 + v^2/2$.

The zeroth component of the four acceleration is

$$\alpha_0 = \frac{du^0}{dt} = \frac{d}{dt}\left(1 + \frac{v^2}{2}\right) = v\dot{v}$$

The spatial components are

$$\alpha_i = \frac{du^i}{dt} = \frac{d}{dt}\left[\left(1 + \frac{v^2}{2}\right)v^i\right] = (v\dot{v})v^i + \dot{v}\left(1 + \frac{v^2}{2}\right) \approx \dot{v}$$

so that $\alpha_0 = v\dot{v} \ll \alpha_i = \dot{v}$

We can thus consider the four-acceleration as being invariant under the above change of frame. Using the expression of the four-acceleration in the freely-falling frame, and the relation between the velocities, we find

$$A^i = \alpha^i = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \partial^i\Phi_{\mathrm{gw}} = \frac{\mathrm{d}V^i}{\mathrm{d}T} + g^i + \partial^i\Phi_{\mathrm{gw}} \ ,$$

and since the equation of motion of the particle is $mA^\alpha = F^\alpha$, where F^α is the non-gravitational four-force, we conclude that

$$\frac{\mathrm{d}V^i}{\mathrm{d}T} = -g^i - \partial^i\Phi_{\mathrm{gw}} + \frac{F^i}{m} \ .$$

Simply put, in Newtonian terms, the particle is experiencing the superposition of a gravitational force due to the Earth (which somehow is fictitious, because it is due to the fact that the terrestrial frame is not freely falling), a force due to the gravitational wave, and any other non-gravitational force possibly acting upon it.

Q9. Show that, for small perturbations of the positions of the mirror compared to their equilibrium state, we have

$$\ddot{\delta} + \omega_0^2 \delta = 2\omega^2 h_+ L_0 \cos \omega t, \tag{10}$$

where you will specify the expression of ω_0, L_0.

Solution Q9

In this case, the mirrors are suspended in such a way that they can move slightly in response to gravitational waves, similar to how a pendulum can swing from its equilibrium position. When they are slightly pushed away from their equilibrium position, and in the absence of any further perturbation, they would start oscillating with a frequency $\omega_0/(2\pi)$, where $\omega_0 = \sqrt{g/\ell}$ and ℓ is the length of the suspension. The classical treatment of the motion of a pendulum with small amplitude leads to

$$\ddot{X} = -\omega_0^2(X - X_0) + \frac{1}{2}\,\omega^2 h_+ X \cos(\omega t),$$

where we considered that $z = 0$ in the phase of the gravitational wave.

Proof of : $\ddot{X} = -\omega_0^2(X - X_0) + \frac{1}{2}\,\omega^2 h_+ X \cos(\omega t)$

We need to consider two separate effects on the motion of the mirrors in a gravitational wave interferometer: the natural oscillatory motion of the mirror (as it behaves like a pendulum) and the additional effect of the gravitational wave, so

$$\ddot{X} = F_{\text{pd}} + F_{\text{gw}}$$

For small displacements, the motion of the pendulum (mirror) can be approximated as simple harmonic motion, therefore

$$F_{\text{pd}} = -\omega_0^2(X - X_0)$$

On the other hand for F_{gw} we can write

$$\vec{F}_{\text{gw}} = -\nabla \Phi_{\text{gw}}$$

The X-component of the force is given by the partial derivative of the potential with respect to X (note that we have $y = z = 0$ in the equation (7)),

$$\begin{aligned}
F_x = -\frac{\partial \Phi_{\text{gw}}}{\partial X} &= -\frac{\partial}{\partial X}\left(\frac{\omega^2}{8}[h_+(-X^2)]e^{-i\omega t} + \text{c.c.}\right) \\
&= \frac{\omega^2}{4} h_+ X (e^{-i\omega t} + \text{c.c.}) \\
&= \frac{1}{2}\omega^2 h_+ X \cos(\omega t)
\end{aligned}$$

where in the last step we used $e^{-i\omega t} + e^{i\omega t} = 2\cos(\omega t)$.

Similarly, for Y,

$$\ddot{Y} = -\omega_0^2(Y - Y_0) - \frac{1}{2}\,\omega^2 h_+ Y \cos(\omega t).$$

By subtraction, we finally obtain

$$\ddot{\delta} + \omega_0^2 \delta = \omega^2 h_+ (X + Y)\cos(\omega t) \approx 2\omega^2 h_+ L_0 \cos(\omega t),$$

where $\ddot{\delta} = 2(\ddot{X} - \ddot{Y})$ and $L_0 = (X_0 + Y_0)/2$. As we can see, this is a forced harmonic oscillator equation, where δ is the oscillation displacement, ω_0 is the natural frequency of the oscillator, ω

is the frequency of the gravitational wave, h_+ is the amplitude of the gravitational wave, and L_0 is the average equilibrium position of the interferometer arms.

Q10. What is the resulting amplitude of the oscillations of δ? Given that the height of LIGO's suspensions is on the order of 1 m, and that the typical frequency of the observed gravitational waves is 100 Hz, in which regime is the oscillator?

Solution Q10

We have seen:

$$\ddot{\delta} + \omega_0^2 \delta = F_0 \cos(\omega t),$$

where $F_0 = 2\omega^2 h_+ L_0$ is the amplitude of the driving force. One solution to the above equation is given by

$$\delta = A\cos(\omega t)$$

plugging this back into the equation, we can solve for the amplitude A

$$A = \delta_{\text{max}} = \frac{2\omega^2\, h_+ L_0}{|\omega_0^2 - \omega^2|}$$

Since $\omega_0 = \sqrt{\frac{g}{\ell}} \sim 3.18$ rad/s ~ 0.5 Hz, we are in the regime $\omega \gg \omega_0$, so that $\delta_{\text{max}} \approx 2h_+ L_0$.

Q11. Given that the frequency of the laser of LIGO is $\lambda_{\text{l}} = 1\,\mu\text{m}$, that the effective[1] length of the arms is $L_0 = 1.2 \times 10^6$ m, and that the strain amplitude of the first wave detected was $h_+ = 10^{-21}$, deduce the sensitivity of the LIGO instrument with respect to changes of luminous intensity.

Solution Q11

To find the change in intensity as a function of a small change in phase difference, we differentiate the intensity formula (8) with respect to ϕ:

$$\frac{dI}{d\phi} = -\frac{I_0}{4}\sin\phi \approx -\frac{I_0}{4}\phi$$

where we assumed ϕ is small. For a maximal change in path length δ_{max}, the phase difference is

$$\phi = \frac{2\pi}{\lambda_{\text{l}}}\delta_{\text{max}}$$

therefore the relative changes in the intensity is given by

$$\frac{|dI/d\phi|}{I_0} \approx \frac{\pi}{2}\frac{\delta_{\text{max}}}{\lambda_{\text{l}}} \approx \pi\frac{h_+ L_0}{\lambda_{\text{l}}} \approx 3.6 \times 10^{-9}.$$

The photodiode of LIGO is therefore sensitive to relative changes of the luminous intensity up to a part in a billion!

Q12. Gravitational waves are often presented as "ripples in spacetime", which "stretch space as they propagate". What do you think about these expressions? Could we detect gravitational waves if the mirrors were standing on the ground?

Solution Q12

The idea of the stretching of space usually comes from the relation $\delta = 2h(t)L_0$. Indeed, everything seems to happen as if space were expanding in a direction, while contracting in the orthogonal direction. However, the path to this relation is paved with assumptions. In particular, it relies

[1] LIGO contains a resonant cavity that essentially multiplies the 4 km-arms by a factor 300.

on the fact that the mirrors essentially in horizontal free fall—as far as $\omega \gg \omega_0$—in the sense that no forces apart from gravitation act on them.

If the mirrors of the interferometer did not have this quasi-freedom of horizontal motion, then δ would almost not change, and hence it would be impossible to detect gravitational waves with such an interferometer. Therefore, it is much more natural to think of gravitational waves as *propagating tidal forces*, i.e. propagating curvature, instead of calling any vague notion of stretching of space.