Prelude to Vectors, Forms, and Tensors

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1 Electromagnetic field

Note. In this exercise, we will consider *inertial* coordinates only, for which the metric takes the standard Minkowski form $[\eta_{\mu\nu}] = [\eta^{\mu\nu}] = \text{diag}(-1, 1, 1, 1)$. You will assume that this metric can be used to raise and lower indices, in the sense that $A^\mu = \eta^{\mu\nu} A_\nu$ and $A_\mu = \eta_{\mu\nu} A^\nu$. Do not worry if this seems obscure, it will be extensively discussed later in the course.

In relativistic electrodynamics, the fundamental field is a one-form $\boldsymbol{A} = A_\mu \mathbf{d}x^\mu$ associated with the scalar and vector potentials of electromagnetism, as $(A_\mu) = (-V, \vec{A})$. We then define the following quantity:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]}. \tag{1}$$

Q1. How does $F_{\mu\nu}$ transform under a general coordinate transformation $\{x^\mu\} \to \{y^\alpha\}$. What is the geometric status of this quantity?

Solution Q1

Since A_μ are the components of a form, we know that they transform as

$$A_\alpha = A_\mu \frac{\partial x^\mu}{\partial y^\alpha};$$

thus

$$F_{\alpha\beta} \equiv 2\partial_{[\alpha} A_{\beta]} = 2 \frac{\partial}{\partial y^{[\alpha}} \left(A_\nu \frac{\partial x^\nu}{\partial y^{\beta]}} \right) = 2A_\nu \frac{\partial^2 x^\nu}{\partial y^{[\alpha} \partial y^{\beta]}} + 2 \frac{\partial A_\nu}{\partial y^{[\alpha}} \frac{\partial x^\nu}{\partial y^{\beta]}} = 2\partial_\mu A_\nu \frac{\partial x^\mu}{\partial y^{[\alpha}} \frac{\partial x^\nu}{\partial y^{\beta]}}.$$

Now modulo a relabelling of the mute indices, we find that the antisymmetrisation over α, β is equivalent to an antisymmetrisation over μ, ν, indeed:

$$\begin{aligned}
2\partial_\mu A_\nu \frac{\partial x^\mu}{\partial y^{[\alpha}} \frac{\partial x^\nu}{\partial y^{\beta]}} &= \partial_\mu A_\nu \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} - \partial_\mu A_\nu \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \\
&= \partial_\mu A_\nu \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} - \partial_\nu A_\mu \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \\
&= 2\partial_{[\mu} A_{\nu]} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}.
\end{aligned}$$

Whence

$$F_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} F_{\mu\nu}.$$

This means that the $F_{\mu\nu}$ defined as in eq. (1) are the components of a tensor. It is called the *Faraday tensor*, or *field strength tensor*.

Recall that, in standard electromagnetism, the relations between the electromagnetic field and the electromagnetic potentials are

$$E_i = -\partial_i V - \partial_t A_i, \tag{2}$$
$$B^i = \varepsilon^{ijk}\partial_j A_k, \tag{3}$$

where ε^{ijk} is the three-dimensional permutation symbol.

Q2. Express the components E_i and B^i of the electromagnetic field as a function of $F_{\mu\nu}$. Extending the components of the electric and magnetic fields to four dimensions, assuming $E^0 = B^0 = 0$, show that

$$E^\mu = F^{\mu\nu}u_\nu, \tag{4}$$
$$B^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}, \tag{5}$$

where $\boldsymbol{u}$ is the four-velocity of an observer at rest with respect to the coordinate system (t, x^i). The symbol $\varepsilon_{\mu\nu\rho\sigma}$ denotes the four-dimensional permutation symbol, with the convention $\varepsilon^{0ijk} = -\varepsilon^{ijk}$. What is the advantage of eqs. (4), (5) over eqs. (2), (3)?

Solution Q2

It is straightforward to check that

$$E_i = \partial_i A_0 - \partial_0 A_i = F_{i0}$$
$$B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$$

In his own frame, the observer is only moving in time, not in space (his spatial velocity with respect to himself is zero). The components of his four-velocity $\boldsymbol{u}$ are therefore $(u^\mu) = (1, 0, 0, 0)$ and $(u_\mu) = (-1, 0, 0, 0)$. In other words $u_\mu = -\delta^0_\mu$, so that the contraction of u_μ with any tensor 'selects' its component for $\mu = 0$ and negates it. As a result

$$F^{\mu\nu}u_\nu = -F^{\mu 0} = \begin{cases} F^{00} = 0 = E^0 & \text{for } \mu = 0, \\ -F^{i0} = F^i{}_0 = F_{i0} = E^i & \text{for } \mu = i. \end{cases}$$

Similarly,

$$-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu 0\rho\sigma}F_{\rho\sigma} = -\frac{1}{2}\varepsilon^{0\mu jk}F_{jk} = \begin{cases} 0 = B^0 & \text{for } \mu = 0, \\ \frac{1}{2}\varepsilon^{ijk}F_{jk} = B^i & \text{for } \mu = i. \end{cases}$$

The advantage of the relations (4), (5) is that they only involve four-dimensional vectors and tensors, thus they are coordinate independent, or *covariant*.

Q3. Suppose that another observer, with a different four-velocity $\boldsymbol{v} \neq \boldsymbol{u}$, performs measurements of this electromagnetic field. Does she obtain the same electric and magnetic fields?

Solution Q3

It is now clear that the separation between electric/magnetic fields is frame-dependent. Indeed, the observer with four-velocity $\boldsymbol{v}$ would call $\boldsymbol{E}$ and $\boldsymbol{B}$ the quantities

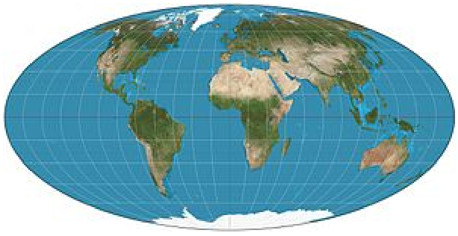
$$E^\mu_{(\boldsymbol{v})} = F^{\mu\nu}v_\nu \neq F^{\mu\nu}u_\nu,$$
$$B^\mu_{(\boldsymbol{v})} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}v_\nu F_{\rho\sigma} \neq -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}.$$

For instance, if the observer with four-velocity $\boldsymbol{u}$ observes, in his own frame, a purely electric field $\boldsymbol{E}_{\boldsymbol{u}}$ (while $\boldsymbol{B}_{(\boldsymbol{u})} = \boldsymbol{0}$), then $F_{\mu\nu} = 2u_{[\mu}E^{(\boldsymbol{u})}_{\nu]}$. But the second observer will generically observe a non-zero magnetic field:

$$B^{\mu}_{(\boldsymbol{v})} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}v_{\nu}F_{\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}v_{\nu}u_{[\rho}E^{(\boldsymbol{u})}_{\sigma]}.$$

In the frame given by $\boldsymbol{u}$, this reads $B^i = -\varepsilon^{ijk}v_j E_k$.

2 Mollweide projection of the sphere



The Mollweide projection is a particular coordinate system which allows one to represent the surface of a sphere on a flat map, where it appears as the interior of an ellipse. Contrary to the Mercator projection (used for most maps of the world), the Mollweide projection better preserves areas but changes angles.

If the sphere has a radius R, a point P on its surface with spherical coordinates $(\theta, \varphi) \in [0, \pi] \times [-\pi, \pi]$ has Mollweide coordinates defined by

$$x = \frac{2\sqrt{2}R}{\pi}\varphi\cos\psi \tag{6}$$

$$y = \sqrt{2}R\sin\psi, \tag{7}$$

where $\psi \in [-\pi/2, \pi/2]$ is an auxiliary angle implicitly defined by $2\psi + \sin 2\psi = \pi\cos\theta$.

Q1. What are the Mollweide coordinates of the poles?

Solution Q1

The North and South poles of the sphere are respectively given by $\theta = 0, \pi$. Introducing this in the equation for ψ, we get $2\psi + \sin 2\psi = \pm\pi$, whence $\psi = \pm\pi/2$. The poles are therefore located on the map at $(x = 0, y = \pm\sqrt{2}R)$.

Q2. Show that the Mollweide components of a vector field $\boldsymbol{v} = v^{\theta}\hat{\boldsymbol{\theta}} + v^{\varphi}\hat{\boldsymbol{\varphi}}$ read

$$v^x = -\frac{xyF(y)}{\sqrt{2R^2 - y^2}}v^{\theta} + \frac{2}{\pi}\sqrt{2R^2 - y^2}\,v^{\varphi}, \tag{8}$$

$$v^y = \sqrt{2R^2 - y^2}\,F(y)\,v^{\theta}, \tag{9}$$

with

$$F(y) \equiv -\frac{\pi}{4}\left(1 - \frac{y^2}{2R^2}\right)^{-1}\sqrt{1 - \frac{4}{\pi^2}\left[\arcsin\left(\frac{y}{\sqrt{2}R}\right) + \frac{y}{\sqrt{2}R}\sqrt{1 - \frac{y^2}{2R^2}}\right]^2} \tag{10}$$

Solution Q2

The Mollweide components $(v^a) = (v^x, v^y)$ of a vector $\boldsymbol{v}$ are obtained from the spherical components $(v^{\alpha}) = (v^{\theta}, v^{\varphi})$ via $v^a = (\partial x^a/\partial\theta^{\alpha})v^{\alpha}$, that is,

$$v^x = \frac{\partial x}{\partial\theta}v^{\theta} + \frac{\partial x}{\partial\varphi}v^{\varphi}$$

$$v^y = \frac{\partial y}{\partial\theta}v^{\theta} + \frac{\partial y}{\partial\varphi}v^{\varphi}.$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial x}{\partial \theta} &= -\frac{2\sqrt{2}R}{\pi}\varphi\frac{\partial \psi}{\partial \theta}\sin\psi = -x\frac{\partial \psi}{\partial \theta}\tan\psi \\
\frac{\partial x}{\partial \varphi} &= \frac{2\sqrt{2}R}{\pi}\cos\psi = \frac{2}{\pi}\sqrt{2R^2-y^2} \\
\frac{\partial y}{\partial \theta} &= \sqrt{2}R\frac{\partial \psi}{\partial \theta}\cos\psi = \frac{\partial \psi}{\partial \theta}\sqrt{2R^2-y^2} \\
\frac{\partial y}{\partial \varphi} &= 0.
\end{aligned}$$

Since $\sin\psi = y/(\sqrt{2}R)$ we find directly $\tan\psi = y/\sqrt{2R^2-y^2}$. Besides, taking the derivative of the implicit definition of ψ with respect to θ yields

$$\begin{aligned}
\frac{\partial \psi}{\partial \theta} &= -\frac{\pi}{2}\frac{\sin\theta}{1+\cos 2\psi} \\
&= -\frac{\pi}{4}\frac{\sqrt{1-\left(\frac{2\psi+\sin 2\psi}{\pi}\right)^2}}{1-\sin^2\psi} \\
&= -\frac{\pi}{4}\left(1-\frac{y^2}{2R^2}\right)^{-1}\sqrt{1-\frac{4}{\pi^2}\left[\arcsin\left(\frac{y}{\sqrt{2}R}\right)+\frac{y}{\sqrt{2}R}\sqrt{1-\frac{y^2}{2R^2}}\right]^2} \\
&\equiv F(y).
\end{aligned}$$

whence the final result.

Q3. Consider winds whose velocity pattern on the globe reads $\boldsymbol{v} = v_0\hat{\boldsymbol{\varphi}}$, where v_0 is a constant. How does it appear on the Mollweide projection?

Solution Q3

For $\boldsymbol{v} = v_0\hat{\boldsymbol{\varphi}}$ we find directly

$$\boldsymbol{v} = \frac{2v_0}{\pi}\sqrt{2R^2-y^2}\,\hat{\boldsymbol{x}}$$

which appears on the map as a purely horizontal field, whose amplitude is homogeneous along $y = \text{cst}$ lines, is maximum at $y = 0$ and vanishes on the poles.