Weak Gravitational Fields

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

1 Heuristic approach to the Newtonian metric

You have demonstrated in the lectures that, for weak gravitational fields, the metric reads at lowest order

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + (1-2\Phi)\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{1}$$

where Φ is the Newtonian gravitational potential. What does not trivially come out of the rigorous derivation of the above line-element is the physical meaning of the global coordinates (t, x^i). In particular, they are *not* some Riemann of Fermi normal coordinates, and they do not coincide with the times and distances measured by a static observer ($x^i = \mathrm{cst}$) in that space-time.

This short exercise aims to determine the nature of (t, x^i) from a *Gedankenexperiment* and simple arguments following from Einstein's equivalence principle. For simplicity, we will restrict to a one-dimensional situation, and focus on the coordinates (t, x)

Consider the following situation (see fig. 1): an experimentalist, Alice, goes on the top of a high building made of many identical floors of thickness h. On her way up, she leaves on each floor a clock ticking every second; she keeps a clock for herself and a ruler. We take the top of the building as the zero-point of the Newtonian potential: $\Phi_{\mathrm{top}} = 0$. Let E be an arbitrary event happening in the building—e.g., a door slams. We define its coordinates t, x as follows: Alice jumps from the top of the building so that she reaches the door's floor at the moment when it slams; t is defined as the time indicated on her own clock, and x is the distance that she has fallen *as measured in her own frame.*

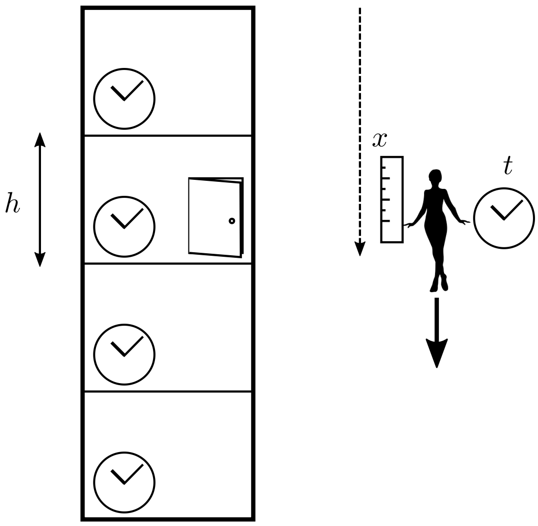

Figure 1 The coordinates x, t are defined by the freely-falling observer.

Q1. Let two events E_1, E_2 be the successive ticks ($\delta\tau = 1\,\mathrm{s}$) of a clock on a floor where the potential is Φ. Using only special relativity and Newtonian dynamics, show that

$$\delta t \equiv t_2 - t_1 = \frac{\delta\tau}{\sqrt{1+2\Phi}}, \tag{2}$$

and conclude that $g_{tt} = -(1+2\Phi)$.

Solution Q1

In Alice's frame, the two events are separated by a distance $d = v\delta t$, where v is the velocity of Alice when she reaches the relevant floor. Since she is freely falling, the metric in her frame is Minkowskian, and hence the space-time interval between E_1, E_2 is

$$\delta s^2 = -\delta t^2 + d^2 = -\delta t^2(1-v^2) = -\delta t^2(1+2\Phi) \ ,$$

where we used Newtonian dynamics in the last equality. Besides, since E_1 and E_2 belong to the same floor, we have $x_1 = x_2$, and thus $\delta s^2 = g_{tt}\delta t^2 = -\delta\tau^2$. Therefore,

$$\delta t = \delta\tau/\sqrt{1+2\Phi}\ , \qquad g_{tt} = -(1+2\Phi)\ .$$

Q2. Show that the thickness of this floor as measured by Alice is

$$\delta x = h\sqrt{1+2\Phi}\ , \tag{3}$$

and deduce from it that $g_{xx} = (1+2\Phi)^{-1} \approx (1-2\Phi)$. On Earth, what is the order of magnitude of the mistake that we make with this latter approximation? [Hint: Use that Alice measures the distance between two doors which appear to slam at the exact same moment in her frame.]

Solution Q2

When Alice measures the thickness of the floor (i.e. the distance between two floors), she performs a measurement for two simultaneous events in her own frame, e.g. the distance between two doors which appear to slam at the exact same moment in her frame. She thus measures a contracted length

$$\delta x = h\sqrt{1-v^2} = h\sqrt{1+2\Phi}\ ,$$

as h is the proper thickness of the floor. Besides, since $\delta t = 0$, we have $h^2 = g_{xx}\delta x^2$, whence $g_{xx} = (1+2\Phi)^{-1} = 1 - 2\Phi + \mathcal{O}(\Phi^2)$. On Earth, we can estimate the potential as (do not forget the c factors!)

$$|\Phi| = \frac{GM_\oplus}{R_\oplus c^2} = \frac{6.7\times10^{-11}\, 6.0\times10^{24}}{6.3\times10^6\,(3.0\times10^8)^2} = 7.1\times10^{-10}\ ,$$

hence we make a mistake on the order of $\approx 5\times10^{-19}$ when neglecting Φ^2 terms.

Q3. Considering the emission and reception of light from the point of view of a static observer in the building, show simply that $g_{tx} = 0$.

Solution Q3

Let an observer at a given floor in the building send light to the ceiling and measure the duration $\delta\tau$ of this travel. We have seen that the static clock ticks slower than t, with $\delta t = \delta\tau/\sqrt{1+2\Phi}$. Besides, the floor and ceiling are separated by a coordinate distance $\delta x = h\sqrt{1+2\Phi}$. Since light always propagates at the speed of light, we also have

$$1 = \frac{h}{\delta\tau} = \frac{\delta x}{\delta t} \times (1+2\Phi)^{-1}\ .$$

Since emission and reception are separated by a null interval, we also have $\delta s^2 = 0$, hence

$$\begin{aligned}
0 = \delta s^2 &= g_{\mu\nu}\delta x^\mu \delta x^\nu \\
&= -(1+2\Phi)\delta t^2 + 2g_{tx}\delta t\delta x + (1+2\Phi)^{-1}\delta x^2 \\
&= -(1+2\Phi)^{-1}\delta x^2 + 2g_{tx}(1+2\Phi)^{-1}\delta x^2 + (1+2\Phi)^{-1}\delta x^2 \\
&= 2g_{tx}h^2
\end{aligned}$$

from which we conclude that $g_{tx} = 0$, so that

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

up to second-order terms in Φ.

Important remark. One could be worried about the fact that $\delta x/\delta t \neq 1$ above. After all, t and x are quantities that are defined from Alice's measurements, so does it mean that light does not propagate at the speed of light in Alice's frame? Fortunately it does not. The point is that δt and δx are *not* the coordinate intervals between light emission and light reception in Alice's frame. They are merely labels, attached to the building, which specify fake distances and fake clock ticking rates. To see that more concretely, remember that δx was previously defined as the distance between two simultaneous events in Alice's frame; but emission and reception of photons are not simultaneous in her frame; in between, the ceiling has moved up!

2 Gravito-magnetism

This exercise is dedicated to a post-Newtonian feature of relativistic gravitation, namely *gravito-magnetism*. Recall that, if the metric is almost Minkowskian, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, then the linearised Einstein's equation reads,

$$\Box\gamma_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \qquad \gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\, h\, \eta_{\mu\nu} \ , \tag{4}$$

in Hilbert's gauge $\partial_\mu \gamma^{\mu\nu} = 0$. We assume that matter is made of non-relativistic massive objects, which implies that $T_{00} \gg T_{0i} \gg T_{ij}$. In this exercise, we will neglect T_{ij} but keep T_{0i}.

Q1. Introducing $A_\mu \equiv \gamma_{0\mu}/4$ and $J_\mu \equiv G T_{0\mu}$, show that the linearised Einstein equation is analogous to relativistic electrodynamics. What plays the role of the electric charge? electric currents? Discuss the presence of a perhaps unexpected minus sign.

Solution Q1

In terms of the A_μ, J_μ notation, the linearised Einstein equation reads

$$\Box A_\mu = -4\pi J_\mu \ ,$$

which is identical to the equation of motion of special-relativistic electrodynamics in Lorenz gauge and in Gauss's units. The Lorenz gauge condition in electrodynamics is analogous to the Hilbert gauge in the context of linearised gravity, since $\partial_\mu \gamma^{\mu\nu} = 0$ for $\nu = 0$ leads to $\partial_\mu A^\mu = 0$.

Recall that, in electrodynamics, $(J^\mu_{\rm e}) = (\rho_{\rm e}, j^i_{\rm e})$, where $\rho_{\rm e}$ is the electric charge density and $j^i_{\rm e}$ is the electric current density. In the present analogy, we thus have

$$\rho_{\rm e} = J^0_{\rm e} \leftrightarrow J^0 = G T^0_0 = -G\rho \ ,$$
$$j^i_{\rm e} = J^i_{\rm e} \leftrightarrow J^i = G T^i_0 = -G P^i \ ,$$

where ρ is the energy density and P^i the momentum density, i.e. the energy flux density. The equivalent of electric charge is thus $-Gm$ in this analogy. The presence of a minus sign translates the fact that positive masses attract each other, while in electrodynamics positive charges repel each other.

Remark. There is a certain freedom in the way electrodynamic quantities are associated with gravitational quantities. For instance, it could seem more natural to directly associate electric charge with mass, and include $-G$ in the coupling constant. Working with SI units instead of Gauss units, we would then have $J^{\rm m}_\mu \equiv -T_{0\mu}$, and $\mu_{\rm g} \equiv -4\pi G$, so that

$$\Box A_\mu = -\mu_{\rm g} J^{\rm m}_\mu \ ,$$

where $\mu_{\rm g}$ is here the analogue of magnetic permeability. In that formulation, the attractive character of gravitation is contained in the negativity of $\mu_{\rm g}$, and of $\varepsilon_{\rm g} = 1/\mu_{\rm g}$.

Q2. Show that, at this post-Newtonian order, the metric reads

$$\mathrm{d}s^2 = -(1-2A_0)\mathrm{d}t^2 + 8A_i\mathrm{d}x^i\mathrm{d}t + (1+2A_0)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{5}$$

Solution Q2

Taking the trace of $\gamma_{\mu\nu}$ in eq. (4) we get

$$\gamma \equiv \eta^{\mu\nu}\gamma_{\mu\nu} = h - 2h = -h$$

The metric perturbation $h_{\mu\nu}$ is then obtained from $\gamma_{\mu\nu}$ by inverting eq. (4),

$$h_{\mu\nu} = \gamma_{\mu\nu} + \frac{1}{2}h\,\eta_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\gamma\,\eta_{\mu\nu} \ .$$

With $\gamma_{0\mu} = 4A_\mu$ and $\gamma_{ij} = 0$, we have $\gamma = -\gamma_{00}$, hence

$$\begin{aligned} h_{00} &= \gamma_{00} + \frac{1}{2}\gamma = \frac{1}{2}\gamma_{00} = 2A_0 \\ h_{0i} &= \gamma_{0i} = 4A_i \\ h_{ij} &= -\frac{1}{2}\gamma\,\eta_{ij} = 2A_0\,\delta_{ij} \end{aligned}$$

that is to say

$$\mathrm{d}s^2 = -(1-2A_0)\mathrm{d}t^2 + 8A_i\mathrm{d}x^i\mathrm{d}t + (1+2A_0)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ .$$

In the post-Newtonian expansion scheme, the small parameter is the typical velocity of sources, $\varepsilon \sim v$. We also assume that the source of the gravitational fields is 'almost' stationary, i.e. the time derivatives are much smaller than spatial derivatives so that $\partial_t \sim \varepsilon\partial_i$.

Q3. Expanding the geodesic equation at linear order in the metric and in ε, show that freely-falling particles satisfy the following equation of motion:

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = E^i_{\mathrm{g}} + \varepsilon^i{}_{jk}v^jB^k_{\mathrm{g}} \ , \tag{6}$$

with $v^i = \mathrm{d}x^i/\mathrm{d}t$, $v^2 \equiv \delta_{ij}v^iv^j$, and where you will give the expression of $E^i_{\mathrm{g}}, B^i_{\mathrm{g}}$ in terms of A_μ. What is the difference with electrodynamics?

[N.B. This exercise has been slightly modified/simplified, the original formulation required to go up to 2nd order in ε.]

Solution Q3

We start from the geodesic equation under the form

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\rho}u^\nu u^\rho = 0 \ ,$$

with $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau = u^0\mathrm{d}x^\mu/\mathrm{d}t$. Let us first write down the Christoffel symbols as a function of A_μ. At linear order in the metric, we note that $\Gamma^\mu{}_{\nu\rho} = \eta^{\mu\sigma}\Gamma_{\sigma\nu\rho}$ where

$$\begin{aligned} \Gamma_{\sigma\nu\rho} &= g_{\sigma\mu}\Gamma^\mu_{\nu\rho} \\ &= \frac{1}{2}g_{\sigma\mu}g^{\mu\lambda}\left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}\right) \\ &= \frac{1}{2}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right) \end{aligned}$$

For the metric (5) we have: $g_{00} = -(1-2A_0)$, $g_{0i} = g_{i0} = 4A_i$ and $g_{ij} = (1+2A_0)\delta_{ij}$. Then we can calculate $\Gamma^{\mu}{}_{\nu\rho} = \eta^{\mu\sigma}\Gamma_{\sigma\nu\rho}$

$$
\begin{aligned}
\Gamma^0{}_{00} &= -\Gamma_{000} = -\frac{1}{2}\left(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) = -\frac{1}{2}\left(\partial_0(-(1-2A_0))\right) = \boxed{-A_{0,0}} \\
\Gamma^0{}_{0i} &= -\Gamma_{00i} = -\frac{1}{2}\left(\partial_0 g_{0i} + \partial_i g_{00} - \partial_0 g_{0i}\right) = -\frac{1}{2}\left(\partial_i(-(1-2A_0))\right) = \boxed{-A_{0,i}} \\
\Gamma^0{}_{ij} &= -\Gamma_{0ij} = -\frac{1}{2}\left(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ji}\right) = -\frac{1}{2}\left(\partial_i(4A_j) + \partial_j(4A_i) - \partial_0((1+2A_0)\delta_{ij})\right) \\
&= -2(A_{i,j} + A_{j,i}) + A_{0,0}\delta_{ij} = \boxed{-4A_{(i,j)} + A_{0,0}\delta_{ij}} \\
\Gamma^i{}_{00} &= \Gamma_{i00} = \frac{1}{2}\left(\partial_0 g_{i0} + \partial_0 g_{i0} - \partial_i g_{00}\right) = \frac{1}{2}\left(2\partial_0(4A_i) - \partial_i(-(1-2A_0))\right) = \boxed{4A_{i,0} - A_{0,i}} \\
\Gamma^i{}_{0j} &= \Gamma_{i0j} = \frac{1}{2}\left(\partial_0 g_{ij} + \partial_j g_{0i} - \partial_i g_{0j}\right) = \frac{1}{2}\left(\partial_0((1+2A_0)\delta_{ij}) + \partial_j(4A_i) - \partial_i(4A_j)\right) \\
&= A_{0,0}\delta_{ij} + 2A_{i,j} - 2A_{j,i} = \boxed{4A_{[i,j]} + A_{0,0}\delta_{ij}} \\
\Gamma^i{}_{jk} &= \Gamma_{ijk} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) = \frac{1}{2}\left(\partial_j((1+2A_0)\delta_{ik}) + \partial_k((1+2A_0)\delta_{ij}) - \partial_i((1+2A_0)\delta_{jk})\right) \\
&= \boxed{A_{0,j}\delta_{ik} + A_{0,k}\delta_{ij} - A_{0,i}\delta_{jk}}
\end{aligned}
$$

with the convention $T_{(ij)} = (T_{ij} + T_{ji})/2$ and $T_{[ij]} = (T_{ij} - T_{ji})/2$.

Let's consider the spatial part of the geodesic equation. Substituting the Christoffel symbols computed above, we find

$$
\begin{aligned}
\frac{\mathrm{d}u^i}{\mathrm{d}\tau} &= (u^0)^2\left(-4A_{i,0} + A_{0,i}\right) - 2u^0u^j\left(2A_{i,j} - 2A_{j,i} + A_{0,0}\delta_{ij}\right) - 2A_{0,j}u^iu^j + A_{0,i}u^ju^j \\
&= (u^0)^2\left(-4A_{i,0} + A_{0,i}\right) - 4u^0u^j\left(A_{i,j} - A_{j,i}\right) - 2A_{0,0}u^0u^i - 2A_{0,j}u^iu^j + A_{0,i}u^ju^j \\
&= (u^0)^2\left(-4A_{i,0} + A_{0,i}\right) - 4(u^0)^2v^j\left(A_{i,j} - A_{j,i}\right)
\end{aligned}
$$

Note that at linear order in ε ($\varepsilon \sim v$, $\partial_t \sim \varepsilon\partial_i$), we have neglected the following terms

$$
-2A_{0,0}u^0u^i - 2A_{0,j}u^iu^j \approx 0 \qquad , \qquad A_{0,i}u^ju^j \approx 0
$$

On the other hand, considering that $u^i = \gamma v^i = u^0 v^i$, and using the chain rule we can also write

$$
\frac{du^i}{d\tau} = \frac{d(u^0v^i)}{d\tau} = \frac{d(u^0v^i)}{dt}\frac{dt}{d\tau} = u^0\frac{d(u^0v^i)}{dt} = u^0(v^i\frac{du^0}{dt} + u^0\frac{dv^i}{dt}) \approx (u^0)^2\frac{dv^i}{dt}
$$

where in the last step, we neglected the term $u^0v^i\frac{du^0}{dt}$ because it is of order ϵ^2. Therefore, equating the two description of $\frac{du^i}{d\tau}$ and then dividing the whole equation by $(u^0)^2$, we find

$$
\frac{\mathrm{d}v^i}{\mathrm{d}t} = \left[\partial_i A_0 - \partial_0(4A_i)\right] + v^j\left[\partial_i(4A_j) - \partial_j(4A_i)\right] \ , \tag{7}
$$

where the right-hand side is very similar to the Lorentz force. The first term is the electric part, where the "gravito-electric" field is $E^i_{\mathrm{g}} \equiv \partial_i A_0 - \partial_0(4A_i)$, i.e.

$$
\boldsymbol{E}_{\mathrm{g}} \equiv \boldsymbol{\nabla} A_0 - \partial_t(4\boldsymbol{A}) \ ,
$$

to be compared with $\boldsymbol{E} = -\boldsymbol{\nabla}V - \partial_t\boldsymbol{A} = \boldsymbol{\nabla}A_0 - \partial_t\boldsymbol{A}$ in the case of electrodynamics. Thus the only difference here is the presence of a factor 4 in front of A_i. As for the magnetic part, let's consider first the case of electrodynamics:

$$\begin{aligned}\varepsilon^i{}_{jk}\boldsymbol{B}_g^k = \varepsilon^i{}_{jk}(\nabla\times\boldsymbol{A})^k = &= \varepsilon^i{}_{jk}(\varepsilon^k{}_{lm}\partial_l A_m)\\ &= (\delta^i_l\delta_{jm} - \delta^i_m\delta_{jl})\partial_l A_m\\ &= \partial_i A_j - \partial_j A_i\ ,\end{aligned}$$

In our case, however, by comparing equation (7) with (6), we find

$$\varepsilon^i{}_{jk}\boldsymbol{B}_g^k = [\partial_i(4A_j) - \partial_j(4A_i)]$$

so the "gravito-magnetic" field can be defined as

$$\boldsymbol{B}_{\mathrm{g}} \equiv \boldsymbol{\nabla}\times(4\boldsymbol{A})\ .$$

which again differs from its electrodynamic counterpart by a factor four in front of the vector potential. Summarising, at the present post-Newtonian order, a freely-falling particle undergoes a force similar to the Lorentz force. The electric part is the extension of the Newtonian gravitational force $\boldsymbol{\nabla}A_0 = -\boldsymbol{\nabla}\Phi$, corrected by $-4\partial_t\boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential generated by moving masses. This vector potential also generates the analogue of a magnetic field $\boldsymbol{B}_{\mathrm{g}} = \boldsymbol{\nabla}\times(4\boldsymbol{A})$. The factor 4 affecting the magnetic part shows that gravitation is 4 times more efficient at producing gravito-magnetic effects than its electrodynamic counterpart.

Consider a galaxy of the Milky Way type, which we model as a thick disk of radius $R = 20\,\mathrm{kpc}$, and thickness $h = R/100 = 0.2\,\mathrm{kpc}$. For simplicity, we will assume this disk to be homogeneous, with density $0.1M_\odot/\mathrm{pc}^3$, and rigidly rotating with angular velocity $\Omega = 10^{-15}$ rad/s.

Q4. From your knowledge of electro-statics and magneto-statics, give the integral expressions of the gravito-electric and gravito-magnetic fields created by the galaxy. Integrate numerically those results, and compare the gravito-magnetic force with the gravito-electric (Newtonian) force experienced by a star located at the outskirts of the galaxy.

Solution Q4

Let us start with the gravito-electric part. Since the distribution of matter and velocities are stationary, we have $\partial_t\boldsymbol{A} = 0$, so that $\boldsymbol{E}_{\mathrm{g}} = -\boldsymbol{\nabla}\Phi$ just like in Newtonian gravity. We then know that

$$\boldsymbol{E}_{\mathrm{g}}(\boldsymbol{x}) = -G\int \mathrm{d}^3\boldsymbol{x}'\ \rho(\boldsymbol{x}')\,\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\ .$$

Taking advantage of the axial symmetry of the problem, we can assume that $\boldsymbol{x}$ lies within the Oxz-plane, where O denotes the centre of the galaxy, and z its rotation axis. Then the integral becomes

$$\boldsymbol{E}_{\mathrm{g}}(x,z) = -G\rho\int_0^R \mathrm{d}r'\ r'\int_0^{2\pi}\mathrm{d}\phi'\int_{-h/2}^{h/2}\mathrm{d}z'\ \frac{(x - r'\cos\phi')\boldsymbol{e}_x + (z-z')\boldsymbol{e}_z}{[x^2 + r'^2 - 2xr'\cos\phi' + (z-z')^2]^{3/2}}\ .$$

where we used

$$|\boldsymbol{x}-\boldsymbol{x}'|^3 = \left(\sqrt{(x - r'\cos\phi')^2 + (0 - r'\sin\phi')^2 + (z-z')^2}\right)^3$$

For a star at $r = R, z = 0$, we already know by symmetry that $\boldsymbol{E}_{\mathrm{g}}(R,0) = -E_{\mathrm{g}}\boldsymbol{e}_x$; we can extract the dimensions of the galaxy from the relevant integral by defining $X = r'/R$ and $Z = z'/h$, which yields

$$E_{\mathrm{g}} = G\rho h\times I_E(h/R)$$

$$\text{with}\quad I_E(h/R) \equiv \int_0^1\mathrm{d}X\int_0^{2\pi}\mathrm{d}\phi\int_{-1/2}^{1/2}\mathrm{d}Z\ \frac{X(1 - X\cos\phi)}{[1 + X^2 - 2X\cos\phi + (h/R)^2Z^2]^{3/2}} \approx 12.75$$

for $h/R = 10^{-2}$. The resulting field is then equal to $E_\mathrm{g} = 3.6 \times 10^{-11}$ N/kg.

Let us now turn to the magnetic part. From the analogy with magneto-statics, we exploit the Biot & Savart law, which reads

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\boldsymbol{x}' \, \frac{\boldsymbol{J}(\boldsymbol{x}') \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3}$$

Simply replacing the magnetic permeability μ_0 by $-4\pi G/c^2 \times 4$, where the $1/c^2$ is here for the correct units, and the factor 4 is the one that was discussed previously, hence

$$\begin{aligned}
\boldsymbol{B}_\mathrm{g}(\boldsymbol{x}) &= -\frac{4G}{c^2} \int \mathrm{d}^3\boldsymbol{x}' \, \frac{\rho(\boldsymbol{x}')\boldsymbol{v}(\boldsymbol{x}') \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \\
&= -\frac{G\rho\Omega}{c^2} \int_0^R \mathrm{d}r' \, r' \int_0^{2\pi} \mathrm{d}\phi' \int_{-h/2}^{h/2} \mathrm{d}z' \, \frac{r'(r' - x\cos\phi')\boldsymbol{e}_z - (z - z')\cos\phi' \boldsymbol{e}_x}{[x^2 + r'^2 - 2xr'\cos\phi' + (z - z')^2]^{3/2}} \ ,
\end{aligned}$$

so for $x = R$, $z = 0$, we have $\boldsymbol{B}_\mathrm{g} = B_\mathrm{g}\boldsymbol{e}_z$, with

$$\begin{aligned}
B_\mathrm{g} &= \frac{4G\rho h R\Omega}{c^2} \times I_B(h/R) \\
I_B(h/R) &= \int_0^1 \mathrm{d}X \int_0^{2\pi} \mathrm{d}\phi \int_{-1/2}^{1/2} \mathrm{d}Z \, \frac{X^2(\cos\phi - X)}{[1 + X^2 - 2X\cos\phi + (h/R)^2 Z^2]^{3/2}} \approx 8.8 \ .
\end{aligned}$$

The gravito-magnetic force (per unit mass) is given by $\boldsymbol{F}_\mathrm{gm} = \boldsymbol{v} \times \boldsymbol{B}_\mathrm{g} = R\Omega B_\mathrm{g} \boldsymbol{e}_x$ (outwards force), and its ratio with the gravito-electric force is

$$\frac{F_\mathrm{gm}}{F_\mathrm{ge}} = 4\left(\frac{R\Omega}{c}\right)^2 \frac{I_B}{I_E} = 2.6 \times 10^{-5} \ ,$$

which too small to affect the galactic dynamics. Its contribution is comparable to the relativistic enhancement of the “gravitational charge”, by a factor $(1 + v^2/c^2)$, in the gravito-electric force