

Special Relativity

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

In this series of exercises, and for all the next ones, we apply Einstein’s convention of summation over repeated indices.

1 The twin paradox reloaded

Consider twin sisters, Alice and Brienne, who have lived together for twenty years. For her 20th birthday, Brienne is offered a space journey around the Sun. The purpose of this exercise it to calculate the age of both Alice and Brienne when the latter returns from her trip. We assume everything to happen in the Minkowski spacetime, whose metric, in an inertial coordinate system $(x^\mu) = (ct, x, y, z)$, reads

$$ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + \delta_{ij} dx^i dx^j, \tag{1}$$

where Greek indices run from 0 to 3, while Latin indices run from 1 to 3, and δ_{ij} is the Krönecker symbol. We assume that Alice remains at rest on Earth, assumed to be at the origin of this coordinate system ($x_A^i = 0$), while Brienne follows a circular trajectory around the Sun, whose centre is located at $(R, 0, 0)$, with

$$x_B = R - R \cos(vt/R) \tag{2}$$

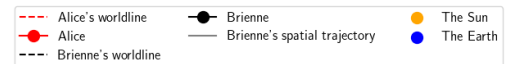
$$y_B = R \sin(vt/R) \tag{3}$$

$$z_B = 0, \tag{4}$$

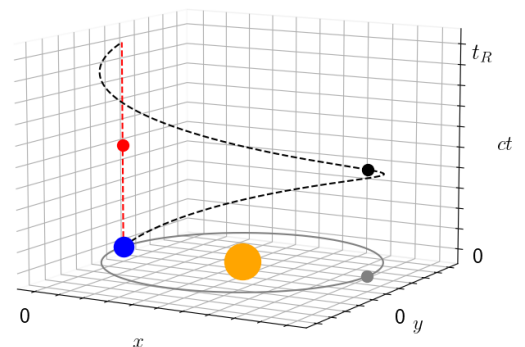
and where v is the velocity of Brienne’s rocket. The event of her departure has coordinates $x^\mu = 0$.

Q1. What are the coordinates of the event corresponding to Brienne’s arrival back on Earth? Sketch the *worldlines* of both Alice and Brienne, on a 3-dimensional diagram where time t is the vertical axis, while x, y span horizontal planes.

Solution Q1



Brienne is back on Earth, $x_{\oplus}^i = 0$, at $t = 2\pi R/v \equiv t_R$, hence the coordinates of the arrival event are $(x^\mu) = (ct_R, 0, 0, 0)$. In this coordinate system, Alice has constant spatial coordinates $x_A^i = 0$, so that her worldline is just a vertical line—she only evolve in time, not in space. Brienne, on the contrary, is not static and moves along a circle. Adding the time dimension, she is found to follow a helicoid in spacetime.



Q2. Suppose that the coordinates of an observer change by the infinitesimal amount dx^μ . At which condition is this displacement said to be *timelike*? In that case, what is the *proper duration* $d\tau$, as measured by the observer's clock, for this displacement?

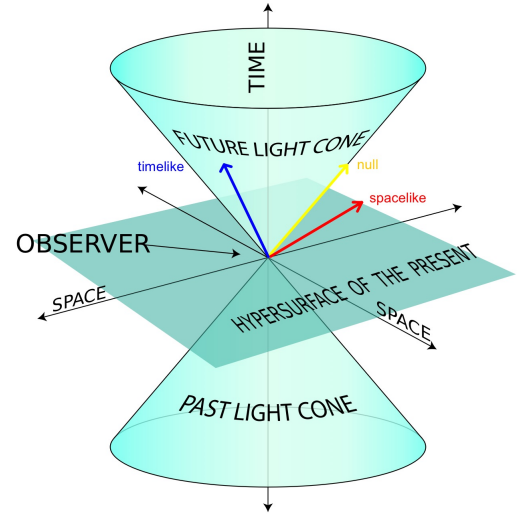
Solution Q2

A displacement dx^μ is said to be timelike if $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu < 0$. In this case we write $ds^2 = -c^2 d\tau^2$, where $d\tau$ is the proper time measured by an observer who is making this displacement, in other words

$$cd\tau = \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \implies d\tau = \sqrt{dt^2 - c^{-2}\delta_{ij}dx^i dx^j}.$$

$$ds^2 = \begin{cases} \text{Timelike:} & ds^2 < 0 \\ \text{Null:} & ds^2 = 0 \\ \text{Spacelike:} & ds^2 > 0 \end{cases}$$

(Diagram credit: Wikipedia, with some notes added.)



Q3. When the time coordinate changes by an amount dt , what is the proper time elapsed for Alice? For Brienne? Calculate the duration of Brienne's trip as measured by Alice, $\Delta\tau_A$, and Brienne, $\Delta\tau_B$. Which one is the shortest? Compute the relative difference $\Delta\tau_A/\Delta\tau_B - 1$ for $v = 0.1c$.

Solution Q3

When t changes by dt , since Alice does not move ($dx_A^i = 0$) we simply find

$$d\tau_A = \sqrt{dt^2 - c^{-2}\delta_{ij}dx_A^i dx_A^j} = dt.$$

Brienne, unlike Alice, has her spatial coordinates changing:

$$\begin{aligned} dx_B &= v \sin(vt/R)dt, \\ dy_B &= v \cos(vt/R)dt, \end{aligned}$$

so that

$$d\tau_B = \sqrt{dt^2 - \frac{v^2}{c^2} \sin^2(vt/R)dt^2 - \frac{v^2}{c^2} \cos^2(vt/R)dt^2} = \sqrt{1 - \frac{v^2}{c^2}} dt$$

and we conclude that

$$\Delta\tau_A = \frac{2\pi R}{v} = \frac{\Delta\tau_B}{\sqrt{1 - (v/c)^2}} > \Delta\tau_B.$$

Brienne therefore always perceives her trip as shorter than Alice. For $v = 0.1c$, we find

$$\frac{\Delta\tau_A - \Delta\tau_B}{\Delta\tau_B} = \frac{1}{\sqrt{1 - (v/c)^2}} - 1 \approx 0.5\%.$$

Now suppose that Brienne has a more complicated trajectory $x^i(t)$; say she swings by Mars and Saturn, stops a bit on the asteroid belt, and finally comes back.

Q4. What is the general expression for the proper duration $\Delta\tau_B$ of Brienne's trip as measured by her own clock? Does she *always* come back younger or older than her sister?

Solution Q4

The proper duration of Brienne's displacement from event E_1 to event E_2 is

$$\tau_B = \int_{E_1}^{E_2} d\tau = \frac{1}{c} \int_{E_1}^{E_2} \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}.$$

This is a curvilinear integral, which depends on the curve which is followed from E_1 to E_2 , i.e. on the motion of the observer connecting those two events. A possibility is to parametrise this curve using coordinate time t . In this case

$$\frac{1}{c} \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \sqrt{1 - c^{-2} \delta_{ij} \frac{dx_B^i}{dt} \frac{dx_B^j}{dt}} dt = \sqrt{1 - \frac{v^2(t)}{c^2}} dt$$

where we have introduced Brienne's *coordinate velocity* v_B . Hence,

$$\tau_B = \int_0^{t_R} dt \sqrt{1 - \frac{v_B^2(t)}{c^2}} < \Delta\tau_A,$$

so that whatever her motion, Brienne always comes back younger than Alice.

2 A dancing particle

Let $\eta_{\mu\nu} = (-1, +1, +1, +1)$ be the Minkowski metric. Let us consider a particle whose coordinates (t, x, y, z) are given by

$$\begin{aligned} t &= f(\lambda) \\ x &= R \cos(\omega\lambda) \\ y &= R \sin(\omega\lambda) \\ z &= c\lambda, \end{aligned}$$

where λ is a parameter with dimensions of time such that $f(0) = 0$ and $f'(\lambda) > 0$, and ω is a frequency with dimensions of inverse time. We define the position 4-vector of the particle as¹

$$x^\mu = (ct, x, y, z).$$

Q1. Compute the 4-velocity of the particle given by

$$u^\mu = \frac{dx^\mu}{d\lambda}.$$

Solution Q1

The 4-velocity can be directly obtained by computing the derivative of the 4-vector x^μ with respect to the parameter λ and is

$$u^\mu = (cf', -R\omega \sin(\omega\lambda), R\omega \cos(\omega\lambda), c).$$

¹Note that there is slight notation abuse as we use x for both the first spatial component and the 4-vector, but there should be no confusion as the 4-vector is always denoted with an index.

Q2. Compute its covariant form $u_\mu \equiv \eta_{\mu\nu}u^\nu$.

Solution Q2

To compute the covariant form of a tensor, we use the matrix multiplication with the metric $\eta_{\mu\nu}$, which gives

$$u_\mu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cf' \\ -R\omega \sin(\omega\lambda) \\ R\omega \cos(\omega\lambda) \\ c \end{pmatrix} = \begin{pmatrix} -cf' \\ -R\omega \sin(\omega\lambda) \\ R\omega \cos(\omega\lambda) \\ c \end{pmatrix}.$$

In this case, thanks to the simple form of the metric, the only effect is to flip the sign of the 0 component.

Q3. Compute the squared norm of the 4-velocity given by $u^2 \equiv u^\mu u_\mu$.

Solution Q3

Again, we use the matrix multiplication to compute the squared norm of the vector, which gives

$$\begin{aligned} u^\mu \eta_{\mu\nu} u^\nu &= \begin{pmatrix} cf' & -R\omega \sin(\omega\lambda) & R\omega \cos(\omega\lambda) & c \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cf' \\ -R\omega \sin(\omega\lambda) \\ R\omega \cos(\omega\lambda) \\ c \end{pmatrix} \\ &= -c^2(f')^2 + (R\omega)^2 + c^2. \end{aligned}$$

Q4. Determine the function $f(\lambda)$ assuming that $u^2 = -c^2$.

Solution Q4

The condition $u^2 = -c^2$ can be inverted to find

$$f'(\lambda) = +\sqrt{2 + \left(\frac{R\omega}{c}\right)^2},$$

where we chose the positive sign as indicated. This relation can be solved directly to give

$$f(\lambda) = \lambda\sqrt{2 + \left(\frac{R\omega}{c}\right)^2},$$

where we used $f(0) = 0$.

Q5. Sketch the trajectory of the particle in a 3D spatial plot. Is the particle massive or massless?

Solution Q5

The trajectory is schematically shown below (with arbitrary units for the various constants). The 4-velocity is timelike as $u^2 < 0$. Hence, the particle must be massive. A massless particle (for example a photon) satisfies $u^2 = 0$.

Note: The condition for being timelike depends on the sign convention for the metric. A useful trick is to look at the sign in front of the dt^2 term in the metric. If this term is negative, then a vector v^μ is timelike if $v^2 < 0$ (and vice-versa).

