WEAK GRAVITATIONAL FIELDS

Most gravitational fields in the universe are weak:

$$\phi = \frac{GM}{Rc^2} \simeq 6.7\cdot 10^{-11}\,\frac{m^3}{kg\,sec^2}\left(\frac{M}{M_\odot}\right) 2\cdot 10^{30}\,kg$$

grav. field by mass M and radius R

$$\left(\frac{R_\odot}{R}\right)\frac{1}{7\cdot 10^8\,m}\,\frac{1}{(3\cdot 10^8)^2} =$$

$$\simeq 2\cdot 10^{-6}\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)$$

→ Earth: $\frac{R_\oplus}{R_\odot} \simeq 10^{-2}$ $\quad \frac{M_\oplus}{M_\odot} \simeq 3\cdot 10^{-6}$ $\quad \phi_{Earth} \simeq 6\cdot 10^{-10}$

→ Sun: $\phi_{Sun} \simeq 2\cdot 10^{-6}$

→ Neutron star: $\frac{M}{M_\odot} \simeq 1.4$ $\quad \frac{R_{NS}}{R_\odot} = \frac{10\,km}{7\cdot 10^5\,km} \simeq 10^{-5}$

$\phi_{NS} \simeq 0.1$ — even for an extreme object like a NS, ϕ is still small

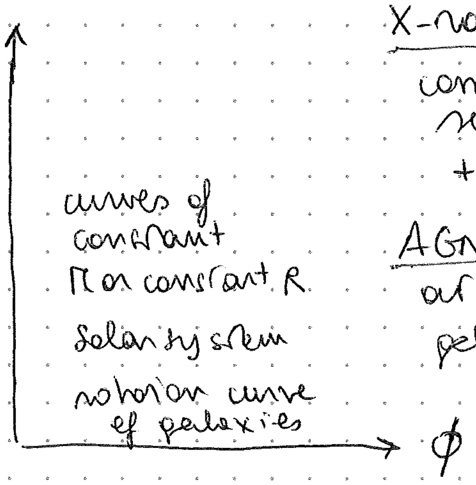
→ BH: $R_s = \frac{GM}{c^2}$ $\quad \phi_{BH} = 1$

Fig 2.1 in the text:

curvature

main sequence: color vs. brightness

$$R_{\mu\nu} \sim \partial\Gamma \sim \partial(g\partial g) \sim \partial^2 g \sim \frac{\phi}{R^2}$$



X-ray binaries: common main sequence star + NS or WD or BH

AGN: massive BH at the centre of a galaxy

IMBH: intermediate mass BH $10^3 - 10^4\,M_\odot$

Since grav. field is weak, one can construct a linearised theory by using perturbation around a bck metric that is the Minkowski one (flat).

To define the linearised theory we need to pick a coordinate system. However GR has diffeomorphisms as a symmetry, and some of this must remain in the new linearised theory: we wouldn't want the theory to be defined only in one special coordinate system. We will explore this and see that sometimes the arbitrariness in the choice of the coordinate system can induce ambiguities, that we call "gauge" issues.

LINEARISED THEORY OF GRAVITY: metric of spacetime is nearly flat, so it exists a coordinate system in which

$\exists$ a coordinate system

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \qquad \text{(ATT! derivatives not small)}$$

We can then expand the field equations and keep only terms that are <u>linear</u> in $h_{\mu\nu}$:

$$(1.68) \qquad \Gamma^{\rho}_{\mu\nu} \simeq \frac{1}{2}\left(\eta^{\rho\sigma} + \cancel{h^{\rho\sigma}}\right)\left(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} - h_{\mu\nu,\sigma}\right)$$

$$(1.153) \qquad R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} + \cancel{\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho}} - \cancel{\Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu}}$$

Indices are raised and lowered with $\eta_{\mu\nu}$, but not those of the metric:

$$\delta_\mu{}^\alpha = g_{\mu\nu} g^{\nu\alpha} \overset{\substack{\text{suppose} \\ \downarrow}}{=} g_{\mu\nu}\eta^{\nu\gamma}\eta^{\alpha\beta} g_{\gamma\beta} = (\eta_{\mu\nu} + h_{\mu\nu})\eta^{\nu\gamma}\eta^{\alpha\beta}(\eta_{\gamma\beta} + h_{\gamma\beta})$$

$$\simeq \eta_\mu{}^\gamma \eta^\alpha{}_\gamma + \eta_\mu{}^\gamma \eta^\alpha{}_\gamma + \eta_\mu{}^\gamma h^\alpha{}_\gamma \neq \delta_\mu{}^\alpha$$

So what is the inverse metric?

$$\delta_\mu^{\ \alpha} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} + \xi^{\nu\alpha}) \simeq \delta_\mu^{\ \alpha} + h_\mu^{\ \alpha} + \xi_\mu^{\ \alpha} \rightarrow \xi_\mu^{\ \alpha} = -h_\mu^{\ \alpha}$$

the inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

Christoffel symbols: $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(h^\alpha_{\ \nu,\mu} + h^\alpha_{\ \mu,\nu} - h_{\mu\nu}^{\ \ \ ,\alpha}\right)$

Ricci tensor: $R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_{\ \nu} + \cancel{\partial_\rho\partial_\nu h^\rho_{\ \mu}} - \Box h_{\mu\nu}\right)$

$(h = h_\rho^{\ \rho} = \eta^{\alpha\beta} h_{\alpha\beta})$ $\qquad - \frac{1}{2}\left(\partial_\nu\partial_\mu h + \cancel{\partial_\nu\partial_\rho h^\rho_{\ \mu}} - \partial_\nu\partial^\rho h_{\mu\rho}\right)$

$$= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_{\ \nu} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\rho h_{\mu\rho}\right)$$

Ricci scalar: $R = \partial_\rho\partial_\mu h^{\rho\mu} - \Box h$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_{\ \nu} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\rho h_{\mu\rho}\right)$$
$$- \frac{1}{2}\eta_{\mu\nu}\left(\partial_\rho\partial_\alpha h^{\rho\alpha} - \Box h\right) = 8\pi G T_{\mu\nu}$$

Einstein field equations:

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\rho\partial_\mu h^\rho_{\ \nu} - \partial_\nu\partial^\rho h_{\mu\rho} + \eta_{\mu\nu}\partial_\rho\partial_\alpha h^{\rho\alpha} - \eta_{\mu\nu}\Box h$$
$$= -16\pi G T_{\mu\nu}$$

The Bianchi identities can be written $\boxed{G^{\mu\nu}_{\ \ \ ,\nu} = 0}$ because
$\nabla(O(h)) \sim \partial(O(h)) + \Gamma O(h) \sim \partial(O(h))$
let's verify the identities:

$$\partial_\nu \Box h^{\mu\nu} + \cancel{\partial_\nu\partial^\nu\partial^\mu h} - \cancel{\partial_\nu\partial_\rho\partial^\mu h^{\rho\nu}} - \partial_\nu\partial^\nu\partial^\rho h^\mu_{\ \rho} + \cancel{\eta^{\mu\nu}\partial_\nu\partial_\rho\partial_\alpha h^{\rho\alpha}}$$
$$\cancel{- \eta^{\mu\nu}\partial_\nu\Box h} = 0$$

This means that $\boxed{\partial_\nu T^{\mu\nu} = 0}$ $\rightarrow$ ~~this means that~~ the energy momentum tensor of the source is conserved, and therefore in the linearised theory the source of the grav. field is not influenced by the field itself.

Example: dust $T_{\mu\nu} = \rho u_\mu u_\nu$

$$\partial_\nu(\rho u^\mu u^\nu) = \underbrace{u^\mu \partial_\nu(\rho u^\nu)}_{=0 \text{ because of } (1.116)} + \rho u^\nu \partial_\nu u^\mu = 0$$

$$u^\mu u^\nu \partial_\nu \rho + u^\mu \rho\, \partial_\nu u^\nu = -\rho\, \partial^\nu u_\nu = 0$$

and therefore $u^\nu \partial_\nu u^\mu = 0$, which is the equation of a straight geodesic: $\frac{du^\mu}{dt} = \frac{d^2x^\mu}{dt^2} = 0$

This means that the motion of the particles isn't affected by the (weak) gravitational field.

Another example we will see later: two compact objects inspiralling around each other emit GWs, but they don't loose energy because of the GW emission.

We now define $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ $\qquad \gamma = h - 2h$

and $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma$ so that in terms of this variable, the field equations become:

$$\Box\gamma_{\mu\nu} \cancel{- \frac{1}{2}\eta_{\mu\nu}\Box\gamma} \cancel{- \partial_\mu\partial_\nu\gamma} - \partial_\rho\partial_\mu\gamma^\rho{}_\nu \cancel{+ \frac{1}{2}\eta^\rho{}_\nu\partial_\rho\partial_\mu\gamma} - \partial_\nu\partial^\rho\gamma_{\mu\rho}$$
$$\cancel{+ \frac{1}{2}\eta_{\mu\rho}\partial_\nu\partial^\rho\gamma} + \eta_{\mu\nu}\partial_\rho\partial_\alpha\gamma^{\rho\alpha} \cancel{- \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\alpha}\partial_\rho\partial_\alpha\gamma} \cancel{+ \eta_{\mu\nu}\Box\gamma} =$$

$$= \Box\gamma_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\alpha\gamma^{\rho\alpha} - \partial_\rho\partial_\mu\gamma^\rho{}_\nu - \partial_\nu\partial^\rho\gamma_{\mu\rho} = -16\pi G T_{\mu\nu}$$

Is it possible to simplify this equation?

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Let us explore the effect of a change of coordinates on the theory.

Example: Lorentz transformation

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu \qquad \Lambda^T \eta \Lambda = \eta$$ minkowski metric invariant

$$\eta_{\alpha\beta} + h'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\eta_{\mu\nu} + h_{\mu\nu})$$

$h'_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu}$ transform as a tensor.

The theory is covariant under Lorentz transformations (and also under Poincaré transformations) but only if $|h'_{\alpha\beta}| \ll 1$ [a condition that is restrictive for boosts].

Let us now consider infinitesimal coordinate transf.:

$$x'^\mu = x^\mu - \xi^\mu(x) \quad \text{with } |\partial_\nu \xi^\mu| = \mathcal{O}(h)$$

$$g_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h'_{\alpha\beta})$$

$$\simeq \eta_{\mu\nu} - \eta_{\mu\beta}\partial_\nu \xi^\beta + h'_{\mu\nu} - \eta_{\alpha\nu}\partial_\mu \xi^\alpha + \mathcal{O}(h'\partial\xi)$$

$$= \eta_{\mu\nu} + h_{\mu\nu}$$

$$\boxed{h_{\mu\nu} = h'_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu} \longrightarrow h = h' - 2\partial_\alpha \xi^\alpha$$

define now $\gamma'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h'$

$$= \gamma_{\mu\nu} + \partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha$$

so that $\partial_\nu \gamma'^{\mu\nu} = \underbrace{\partial_\nu \gamma^{\mu\nu} + \Box \xi^\mu}_{\text{one can always find a } \xi^\mu \text{ that cancels this term}} + \cancel{\partial_\nu \partial^\mu \xi^\nu} - \cancel{\partial^\mu \partial_\alpha \xi^\alpha}$

let us now choose the new coordinate system x'^{α} (that we rename x^{α}). This is called the Hilbert gauge and in this system $\partial_\nu \gamma^{\mu\nu} = 0$ so the eqs. of motion become

$$\Box \gamma_{\mu\nu} = -16\pi G \, T_{\mu\nu}$$

much simpler: a wave equation!

ATT! Not all components of the metric perturbation are physical waves, only 2 corresponding to a massless spin 2 field (next time)

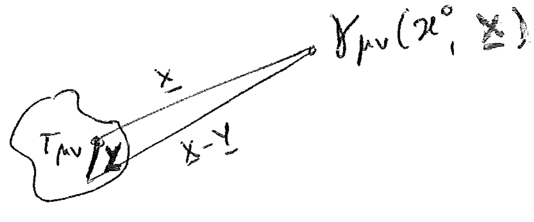
Formal solution: $\gamma_{\mu\nu} = \underbrace{16\pi G \, D_R}_{\text{retarded Green's function}} * T_{\mu\nu} + \underbrace{s_{\mu\nu}}_{\text{solution of the homogeneous eq. (page 9)}}$

$$D_R(x) = \frac{1}{4\pi |\underline{x}|} \delta(x^0 - |\underline{x}|)$$

the retarded solution ($s_{\mu\nu}$ next time - GWs) is

$$\gamma_{\mu\nu} = 4G \int \frac{\delta(x^0 - y^0 - |\underline{x}-\underline{y}|)}{|\underline{x}-\underline{y}|} T_{\mu\nu}(y) \, d^4y = 4G \int \frac{T_{\mu\nu}(x^0 - |\underline{x}-\underline{y}|, \underline{y})}{|\underline{x}-\underline{y}|} d^3y$$

gravitational effects propagate at the speed of light!



gravitational field generated by energy momentum $T_{\mu\nu}$ at time $x_0 - |\underline{x}-\underline{y}|$ in position $\underline{x}$ at time x_0.

<u>Hilbert gauge</u>: $\partial_\nu (D_R * T^{\mu\nu}) = \partial_\nu D_R * T^{\mu\nu} + \cancel{D_R * \partial_\nu T^{\mu\nu}}$

$$= 4G \frac{\partial}{\partial x^\nu} \int (-d^4 z) \frac{\delta(z^0 - |\underline{z}|)}{|\underline{z}|} T^{\mu\nu}(x - z) = -\cancel{D_R * \partial_\nu T^{\mu\nu}} = 0$$

$z = x - y$

NEARLY NEWTONIAN FIELDS

We consider now the Newtonian limit, that is velocities are small, the gravitational field is static, stresses are small as well, and we neglect pressure

$$T_{00} \gg |T_{0i}|, |T_{ij}|$$

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) \qquad u^\mu \simeq (1, \underline{v}) \qquad v \ll 1$$

$$\rho \gg |-(\rho+P)v_i + P g_{0i} + P v_i|, \; |\rho v_i v_j + P g_{ij} + P v_i v_j|$$

From the previous solution for the metric in linearised theory $\bar{\gamma}_{\mu\nu} = 4G \int d^3y \, \frac{T_{\mu\nu}(x^0 - |\underline{x}-\underline{y}|, \underline{y})}{|\underline{x}-\underline{y}|}$

one obtains, with the above assumptions:

$$\bar{\gamma}_{00} = 4G \int d^3y \, \frac{T_{00}(t, \underline{y})}{|\underline{x}-\underline{y}|} = -4\phi, \qquad \bar{\gamma}_{ij} = \bar{\gamma}_{0i} \simeq 0$$

neglect retardation effects because velocities are small

note also that the time dependence in T_{00} is negligible

Therefore, for the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma}$$

$$\begin{cases} g_{00} = -1 - 4\phi + \frac{1}{2} 4\bar{\gamma} = -1 - 2\phi \\ g_{0i} = 0 \\ g_{ij} = \delta_{ij} + \frac{1}{2}\delta_{ij} 4\phi = (1-2\phi)\delta_{ij} \end{cases}$$

since $\bar{\gamma} = \bar{\gamma}^0{}_0 = g^{0\mu}\bar{\gamma}_{\mu 0} = 4\phi$

$$\begin{cases} h_{00} = -2\phi \\ h_{0i} = 0 \\ h_{ij} = -2\phi\,\delta_{ij} \end{cases}$$

so that the line element in this limit can be written

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\phi)\,d\underline{x}^2$$

And in the case in which the source is localized with total mass M and the observer is far away from the source,

$$\phi(\underline{x}) \simeq -G\frac{1}{|\underline{x}|}\int d^3\underline{y}\; T_{00}(t,\underline{y}) = -\frac{GM}{R}$$

$$ds^2 = -\left(1 - 2\frac{GM}{R}\right)dt^2 + \left(1 + 2\frac{GM}{R}\right)d|\underline{x}|^2$$

Newtonian equations of motion

We have seen that in this theory the en. mom. tensor is conserved, so the grav. field does not backreact on the source. However, we can study the behaviour of test particles in the weak gravitational field

Geodesic equation in global inertial coordinates:

$$\frac{d^2x^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

one gets $d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{(1+2\phi)dx^{0\,2} - (1-2\phi)dx^{i\,2}}$

$$\simeq \sqrt{dx^{0\,2} - dx^{i\,2}} \qquad \frac{d\tau^2}{dt^2} = 1 - \frac{dx^{i\,2}}{dt^2} = 1 - v^2 \simeq 1$$

and also $\frac{dx^\mu}{dt} \simeq (1,0,0,0)$

therefore one gets : $\frac{d^2x^0}{dt^2} + \Gamma^0_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad \Gamma^0_{00} = -2\partial_0\phi = 0$

the field is static

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00}\frac{dx^0}{dt} = 0 \qquad \Gamma^i_{00} = -\frac{1}{2}\partial^i h_{00} = -\frac{1}{2}\partial^i(-2\phi) = \partial^i\phi$$

$$\boxed{\frac{d^2x^i}{dt^2} \simeq -\partial^i\phi}$$ the Newtonian equation of motion

Note however that this eq. of motion has been obtained from the geodesic equation, so it gives the trajectory of a free test particle on which no force is acting. Therefore the interpretation of this eq. is very different than in Newtonian theory ~~on~~ within which the grav. field generates a grav. force that accelerate the particle and if the particle had no mass, no accel. is generated and the particle moves on a straight line in the grav. field. In GR, the grav. field curves space-time, and test particle follow geodesic in the curved " ", they are not accelerated. If $\phi \neq 0$, to follow a straight line that is the geodesic of $\eta_{\mu\nu}$, particles need to accelerate.

But this means that in GR massless particles also follow geodesics and light is therefore deflected.

$\boxed{\text{PROPAGATION OF LIGHT RAYS IN GEOMETRIC OPTICS APPR.}}$

Go back to page 41, to define light rays look for oscillating solutions of Maxwell's eqs. 1.186 without source $\nabla^\mu \nabla_\mu A_\nu - \cancel{R^\lambda_\nu A_\lambda} = 0$ assume the scale of variation of A_ν much smaller than curvature

Try $A_\mu = C_\mu e^{iS}$ C_μ slowly varying, S real

$$\nabla^\mu\left[\cancel{\nabla_\mu C_\nu} e^{iS} + C_\nu i e^{iS} \nabla_\mu S\right] \simeq - e^{2iS} \nabla^\mu S \nabla_\mu S +$$

$$+ C_\nu i e^{iS} \nabla^\mu \nabla_\mu S = 0 \Longrightarrow \begin{cases} \nabla^\mu S \nabla_\mu S = 0 & (1) \\ \nabla^\mu \nabla_\mu S = 0 & (2) \end{cases}$$

we now define the <u>wave-vector</u> $k^\mu = \nabla^\mu S$ that is a null vector $k^\mu k_\mu = 0$ from condition (1) and it is also orthogonal to the constant phase surface (S) and also tangent to it from $k^\mu \nabla_\mu S = 0$. (the surface of const phase S is a NULL surface)

From this we get that <u>light rays travel on null geodesics</u>:

$$0 = \nabla_\mu (k^\nu k_\nu) = 2 k^\nu \nabla_\mu k_\nu = 2 k^\nu (\nabla_\mu \nabla_\nu S) =$$

$$= 2 k^\nu (\nabla_\nu \nabla_\mu S) = 2 k^\nu \nabla_\nu k_\mu$$

Lorentz condition $\nabla^\mu A_\mu = 0 = C_\mu i \ldots e^{iS} \nabla^\mu S \rightarrow C_\mu k^\mu = 0$

so the polarization vector C_μ is orthogonal to the wave vector k_μ. Frequency is $\omega = -u^\mu \nabla_\mu S$ u^μ 4-vel of observer

In the <u>nearly-Newtonian limit</u>:

$$(1) \quad g^{\mu\nu} \nabla_\mu S \nabla_\nu S = 0 \qquad -(1-2\phi)(\partial_0 S)^2 + (1+2\phi)(\partial_i S)^2 = 0$$

$g^{\mu\nu} \downarrow \eta^{\mu\nu} - h^{\mu\nu}$; $(\partial_0 S)^2 \downarrow \omega^2$; $(\partial_i S)^2 \downarrow$ only depends on $\underline{x}$ since ϕ does as well

$$g^{00} = -1 + 2\phi$$
$$g^{ij} = \delta^{ij} (1+2\phi)$$

the above eq is satisfied by a function of the form

$$S(x,t) = u(\underline{x}) - \omega t$$

(note, this isn't a plane wave!)

$$-(1-2\phi)\,\omega^2 + (1+2\phi)(\nabla u(\underline{x}))^2 = 0$$

$$(\nabla u(\underline{x}))^2 = \frac{(1-2\phi)}{1+2\phi}\,\omega^2 \simeq (1-4\phi)\,\omega^2$$

light rays propagate as if they were in an inhomogeneous medium with refractive index $n = \sqrt{1-4\phi} \simeq 1-2\phi$ that is spatially varying. They don't follow straight lines!

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GRAVITOMAGNETIC (FIELD) AND LENSE THIRRING EFFECT:

We now study ~~one~~ another deviation from Newtonian dynamics, an effect called "frame dragging": a distribution of mass that is stationary but not static (for example, rotating) causes a rotation of the local inertial system in its vicinity.

We will study this by doing an analogy with the equations of electromagnetism

With respect to nearly-Newtonian case, we keep the terms T_{0i} ($= -\rho v_i$ in the case of the perfect fluid) that are order v so smaller than $T_{00} = \rho$.

field equations: $\begin{cases} \Box \gamma_{0\mu} = -16\pi G\, T_{0\mu} \\ \Box \gamma_{ij} = 0 \end{cases}$ $\Rightarrow$ one index, similar to Maxwell's equations...

define "gravitational vector potential" $A_\mu = \frac{1}{4}\gamma_{0\mu}$
and a "mass current density" $j_\mu = G\,T_{0\mu}$

and with this one has a situation that is equivalent to EM eqs $\Box A_\mu = -4\pi J_\mu$

the condition of stationarity implies $\partial_0 \gamma_{\mu\nu} = 0$

so that $\Delta \gamma_{ij} = 0$

γ_{ij} harmonic function and bounded, Liouville theorem says $\gamma_{ij} = \text{const}$, we put $\gamma_{ij} = 0$ to have reasonable conditions at infinity

This way we only need A_μ to describe the metric completely.

$$A_0 = \frac{\gamma_{00}}{4} \overset{\text{from solution 2-30 that is general}}{=} G \int \frac{T_{00}(\underline{y})}{|\underline{x}-\underline{y}|} d^3y = -\phi$$

$$A_i(\underline{x}) = \frac{\gamma_{0i}}{4} = G \int \frac{T_{0i}(\underline{y})}{|\underline{x}-\underline{y}|} d^3y$$

New term with respect to the Newtonian solution, due to the fact that we didn't neglect T_{0i}

$$\begin{cases} g_{00} = -1-2\phi = -1+2A_0 \\ g_{0i} = \gamma_{0i} = 4A_i \\ g_{ij} = \delta_{ij} - \frac{1}{2}\delta_{ij}\gamma = \delta_{ij}(1+2A_0) \quad \text{(like Newtonian)} \end{cases}$$

$$\gamma = \gamma^\mu_\mu = g^{\mu\nu}\gamma_{\mu\nu} = -\gamma_{00} = -4A_0$$

As in the nearly-Newtonian case, let us now look at the equation of motion for a test particle. We proceed with the analogy, starting from the variational principle:

$$\delta \int \underbrace{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}}_{\mathcal{L}}\, dt \underset{\frac{dx^\mu}{dt}=(1,\underline{v})}{=} \delta \int \sqrt{1-2A_0 \underbrace{-8A_iv^i}_{2g_{i0}u^0u^i} - (1+2A_0)v^2}\, dt$$

This is very similar to the Lagrangian of a charged particle in an external EM field:

$$\mathcal{L}_{EM} = -m\sqrt{1-v^2} - e\phi + e\underline{A}\cdot\underline{v} \simeq -m + m\frac{v^2}{2} - e\phi + e\underline{A}\cdot\underline{v}$$

$$\mathcal{L}_{GM} \simeq 1 - \frac{v^2}{2} + \phi - 4\underline{A}\cdot\underline{v} \quad \text{(Taylor expansions)}$$

- Since the effect is purely gravitational, we need to set $\boxed{e = m}$ to get the analogy
- the sign difference is due to the fact that EM is a repulsive force while the gravitational force is attractive
- the factor 4 difference in $\underline{A} \cdot \underline{v}$ comes partly from g_{0i} and partly from using $\gamma_{\mu\nu}$, a consequence of the tensorial nature of gravity

So to get the eq. of motions we use (1.175) $\left(\begin{array}{l} \underline{E} = \nabla\phi + \frac{\partial A}{\partial t} \\ \underline{B} = \nabla \times \underline{A} \end{array} \right)$

$$u^\mu \partial_\mu u^\nu = \frac{1}{\left(\frac{e}{m}\right)} F^\nu{}_\lambda u^\lambda$$

$$F_{\mu\nu} = 2\, u_{[\mu} E_{\nu]} + \varepsilon_{\mu\nu\alpha} B^\alpha$$

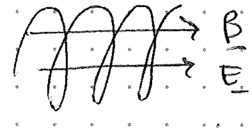
$$\partial_0 v^i + \cancel{v^j \partial_j v^i} = F^i{}_0 + F^i{}_j v_j = E_i + \underline{v} \times \underline{B}$$

and for the gravitomagnetic case it becomes:

$$\ddot{\underline{x}}_i = \underline{E} + 4\, \dot{\underline{x}} \times \underline{B} \qquad (\text{factor } 4)$$

$\nabla A_0 = -\nabla\phi$
effect of the "Newtonian" part of the potential

frame dragging

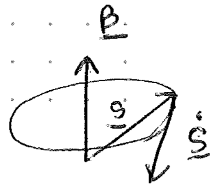


not present in Newtonian gravity

Continuing with the analogy, one can find also an equation that is the equivalent of the spin precession

$$\dot{\underline{S}} = \underline{\mu}_B \times \underline{B} \longrightarrow \frac{4}{2}\, \underline{S} \times \underline{B} = -2\,(\nabla \times \underline{A}) \times \underline{S}$$

magnetic moment $\underline{\mu}_B = \frac{e\,S}{2m} \longrightarrow \frac{S}{2}$



this equation says that a gyroscope in the gravitational field of a rotating mass distribution precesses with angular velocity $\underline{\Omega}_{LT} = -2\, \nabla \times \underline{A}$ where $A_i = \frac{\gamma_{0i}}{4}$

because the rotating mass creates a gravitomagnetic field that drags along a local inertial system

"Lense Thirring effect"

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GRAVITATIONAL WAVES

SOME FACTS:

*) A wave solution of linearized theory

*) GWs emerge naturally in GR because it extends Newtonian theory to give it a causal structure: there is the need of some form of radiation which propagates information at the speed of light.

*) GWs are sourced by accelerated masses, as for EM waves that are sourced by accelerated charges. But the spin nature is very different

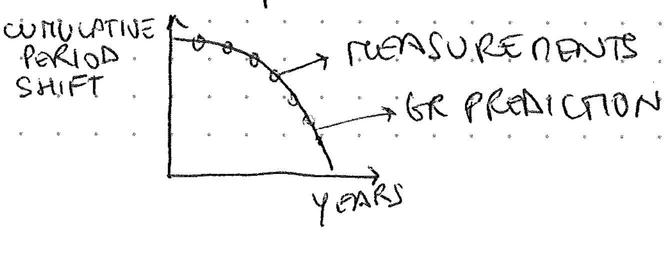
*) the gravitational interaction is weak

→ to produce strong signals, one needs big masses moving almost at the speed of light: essentially astrophysical events that are very energetic processes

→ the detection even of these signals is extremely challenging: it took 50 years from the first detectors to when a detection was finally made: astrophysical events are energetic but far away

*) gauge effects (ie. related to the coordinate choice) create problems for the GW definition: it also took 50 years before the community was convinced that GWs aren't a gauge artefacts but carry energy and momentum

*) the first experimental evidence of the existence of GWs was indirect: the Hulse-Taylor binary pulsar system two NS orbiting around each other and reducing their rotation period because of GW emission



1993 Nobel Prize

*) On the 14/9/2015 the first direct detection was made by the two interferometers LIGO based in USA: the GW emitted by a binary of two black holes in the last phases of their inspiral, and merger

$M_1 = 36 \pm 5\, M_\odot$

$M_2 = 30 \pm 4\, M_\odot$

(sum of R_S) $R_S = 210$ km $\quad (\longrightarrow R_S^\odot = 3$ km$)$

$d_L = 430 \pm 170$ Mpc

*) Since, many detections followed (last catalogue released this summer) including a binary NS in 2017 with an electromagnetic counterpart producing a γ-ray burst

$m_1 = 1.36 - 1.60\, M_\odot$

$m_2 = 1.17 - 1.36\, M_\odot$

$d_L = 40$ Mpc

difference between speed of propagation of GWs and photons can be bounded by the time delay between the arrivals

$$\left|\frac{c_{GW} - c}{c}\right| < 10^{-15}$$

$$\Delta s = c(t_a^\gamma - t_e^\gamma) = c_{GW}(t_a^{GW} - t_e^{GW})$$

$$c(t_a^\gamma - t_e^\gamma) = c_{GW}(t_a^{GW} - t_e^\gamma + \Delta t) \rightarrow \text{delay } \gamma - GW$$

$$\left|\frac{c_{GW} - c}{c}\right| = \left|\frac{c_{GW}}{c} - 1\right| = \left|\frac{t_a^\gamma - t_e^\gamma}{t_a^{GW} - t_e^\gamma + \Delta t} - 1\right| \leq \left|\frac{t_a^\gamma - t_e^\gamma}{t_a^{GW} - t_e^\gamma}\right|$$

$\Delta t \geq 0$

$t_a^{GW} - t_e^\gamma + \Delta t \geq t_a^{GW} - t_e^\gamma$

$$\frac{t_a^\gamma - t_e^\gamma}{t_a^{GW} - t_e^\gamma + \Delta t} \leq \frac{t_a^\gamma - t_e^\gamma}{t_a^{GW} - t_e^\gamma}$$

*) As we have seen, in linearised theory $\partial_\mu T^{\mu\nu} = 0$ so the mass-energy generating the GWs is conserved: there is no back reaction on the orbit due to GW emission. (we will get it with a trick)

the bck metric is flat and the source of $h_{\mu\nu}$ is described by Newtonian gravity.

Let us consider the <u>linearised theory in vacuum</u>

In the Hilbert gauge we have

$$\Box \gamma_{\mu\nu} = 0 \qquad\qquad (\partial_\nu \gamma^{\mu\nu} = 0)$$

In principle, all components of $\gamma_{\mu\nu}$ appear to be radiative. How many propagating degrees of freedom does the theory really have?

$\gamma_{\mu\nu}$ 16 components

$\gamma_{\mu\nu} = \gamma_{\nu\mu} \rightarrow 10$ components left

$\partial_\mu \gamma^{\mu\nu} = 0 \longrightarrow 6$ components left

Are there 6 independent propagating degrees of freedom?

No! GWs have only two independent polarisations, so we should be able to reduce the counting by 4!

This can be done because the Hilbert gauge did not fix the gauge freedom completely: we can still perform another infinitesimal coordinate transformation and remain in the Hilbert gauge. This new coordinate transformation brings us in a new gauge in which we have the additional condition

$$\boxed{\tilde{\gamma} = 0}$$

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new coordinate system $\quad \tilde{x}^\mu = x^\mu - \tilde{\xi}^\mu$

We have seen that $\gamma^{\mu\nu}$ transforms as

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} + \partial_\nu \tilde{\xi}_\mu + \partial_\nu \tilde{\xi}_\mu - \eta_{\mu\nu} \partial_\alpha \tilde{\xi}^\alpha$$

Therefore $\quad \partial_\nu \tilde{\gamma}^{\mu\nu} = \partial_\nu \gamma^{\mu\nu} + \Box \tilde{\xi}^\mu \underset{\uparrow}{\equiv} 0$

we want to remain in the Hilbert gauge

In order to remain in the Hilbert gauge, the coordinate transformation must be such that

$$\boxed{\Box \tilde{\xi}_\mu = 0}$$

Furthermore, we want $\tilde{\gamma} = 0$. The transformation of $\tilde{\gamma}$ is

$$\tilde{\gamma} = \gamma + \partial_\mu \tilde{\xi}^\mu + \partial^\mu \tilde{\xi}_\mu - 4\, \partial_\alpha \tilde{\xi}^\alpha = \gamma - 2\, \partial_\alpha \tilde{\xi}^\alpha$$

Therefore, we also need:

$$\boxed{2\, \partial_\alpha \tilde{\xi}^\alpha = \gamma}$$

Can we find a generic $\tilde{\xi}^\alpha$ that satisfy these conditions? The answer is yes, since $\Box \gamma = 0$: <u>for a scalar γ such that $\Box \gamma = 0$ there always exist a vector $\tilde{\xi}^\mu$ such that $\Box \tilde{\xi}^\mu = 0$ and $\partial_\alpha \tilde{\xi}^\alpha = \frac{\gamma}{2}$</u>. We now construct it.

First let us show : $\quad \Box \gamma_{\mu\nu} = 0 \;\rightarrow\; \Box \gamma = 0$

$$\Box \gamma = \Box \left(g^{\mu\nu} \gamma_{\mu\nu} \right) = \partial_\lambda \big(\underbrace{\cancel{\partial^\lambda g^{\mu\nu}}}_{\text{second order in } h_{\mu\nu}} \gamma_{\mu\nu} + g^{\mu\nu} \partial^\lambda \gamma_{\mu\nu} \big) = g^{\mu\nu} \Box \gamma_{\mu\nu} = 0$$

*) start with a scalar field u satisfying $\Box u = 2\gamma$

*) set $\eta_\mu = \partial_\mu u$ and $\zeta_\mu = \Box \eta_\mu$

*) consequently: $\partial_\mu \zeta^\mu = \Box \partial_\mu \eta^\mu = \Box \Box u = \Box (2\gamma) = 0$

*) Since $\partial_\mu \zeta^\mu = 0$, define an antisymmetric quantity $f^{\mu\nu}$ such that $\quad \zeta^\mu = \partial_\nu f^{\mu\nu}$

$$\left(\partial_\mu \zeta^\mu = \partial_\mu \partial_\nu f^{\mu\nu} = - \partial_\mu \partial_\nu f^{\nu\mu} = - \partial_\nu \partial_\mu f^{\nu\mu} = 0 \right)$$

*) suppose now $f^{\mu\nu} = \Box \sigma^{\mu\nu}$ (always have solutions) and finally set:

$$\tilde{\xi}^{\mu} \equiv \eta^{\mu} - \partial_{\nu}\sigma^{\mu\nu}$$

*) Condition $\Box \tilde{\xi}^{\mu} = 0$:

$$\Box \eta^{\mu} - \Box \partial_{\nu}\sigma^{\mu\nu} = \zeta^{\mu} - \partial_{\nu} f^{\mu\nu} = \zeta^{\mu} - \zeta^{\mu} = 0 \quad \text{ok!}$$

*) Condition $\partial_{\mu}\tilde{\xi}^{\mu} = 2\gamma$:

$$\partial_{\mu}\tilde{\xi}^{\mu} = \partial_{\mu}\eta^{\mu} - \underbrace{\cancel{\partial_{\mu}\partial_{\nu}\sigma^{\mu\nu}}}_{\sigma^{\mu\nu} \text{ as } f^{\mu\nu} \text{ is antisymmetric}} = \Box u = 2\gamma \quad \text{ok!}$$

We therefore found the generic infinitesimal coordinate transformation that allows us to transfer to the TRANSVERSE TRACELESS GAUGE, in which we can impose 4 conditions on the metric $\tilde{\gamma}^{\mu\nu}$ (4 as the components of $\tilde{\xi}^{\mu}$) :

1) $\tilde{\gamma} = 0$ such that $\tilde{\gamma}_{\mu\nu} = \tilde{h}_{\mu\nu}$ and $\tilde{h} = 0$

2) $\tilde{h}^{0i} = 0$ such that $\partial_{\mu}\tilde{h}^{\mu\nu} = \partial_0 \tilde{h}^{00} = 0$ → still in the Hilbert gauge

that also means that in this context we can put $\tilde{h}^{00} = 0$ without loss of generality, because it is a non-dynamical component (the Newtonian potential studied before)

We have therefore that the metric $\tilde{h}_{ij}$ has only <u>two</u> independent components → <u>the physical propagating degrees of freedom</u>.

(From now on we omit the ~)

Equations of motion become : $\Box h_{ij}(\underline{x},t) = 0$
which solutions are plane waves moving at the speed of light

$$h_{ij}(\underline{x},t) = \underbrace{e_{ij}(\underline{k})}_{\text{Polarisation tensors}} e^{ik_\mu x^\mu} \qquad k^\mu = (\omega, \underline{k})$$

$|\underline{k}| = \omega$

null vector

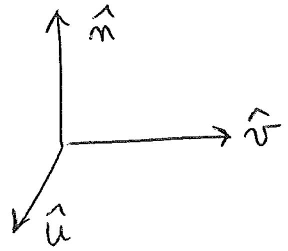
$$k_\mu k^\mu = \eta_{\mu\nu} k^\nu k^\mu = -\omega^2 + |\underline{k}|^2 = 0$$

if $\underline{k} = k\hat{n}$ the tensor $h_{ij}(\underline{x},t)$ has non zero components only in the plane perpendicular to $\hat{n}$:

$$\partial_j h^{ij} = 0 \rightarrow n_j e^{ij} = 0$$

GWs are transverse waves

With two vectors $(\hat{u}, \hat{v})$ orthogonal to $\hat{n}$ and among each other then



$$\begin{cases} e^{+}_{ij} = \hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j \\ e^{\times}_{ij} = \hat{u}_i \hat{v}_j + \hat{u}_j \hat{v}_i \end{cases}$$

a <u>basis</u> for the polarisation tensor

setting $\hat{n} \parallel \hat{z}$, $\hat{u} \parallel \hat{x}$, $\hat{v} \parallel \hat{y}$

$$e^{+}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \qquad e^{\times}_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

and the wave solution becomes then

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$$h_{ij}(z,t) = (h_+ e^{+}_{ij} + h_\times e^{\times}_{ij})\, e^{i(k\hat{n}\cdot\underline{x} - \omega t)}$$

$$= \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{i(kz-\omega t)}$$

the line element of the linearised metric in the TT gauge is:

$$ds^2 = -dt^2 + dz^2 + \left(1 + h_+ e^{i(kz-\omega t)}\right) dx^2 + \left(1 - h_+ e^{i(kz-\omega t)}\right) dy^2 + 2 h_\times e^{i(kz-\omega t)}\, dx\, dy$$

the tensor that projects on the TT gauge a plane wave propagating in the $\hat{n}$ direction and already in the Hilbert gauge

$$h^{TT}_{ij} = \Lambda_{ijkl} h_{kl}$$

$P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j \implies \Lambda_{ijkl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$

projector projecting perpendicular to $\hat{n}$

$P_{ij} \hat{n}^j = 0$

$P_{ij} P_{jk} = P_{ik}$

Properties: ·) $P_{ii} = 0$ but $\Lambda_{iikl} = \Lambda_{ijkk} = 0$ is traceless

·) $\Lambda_{ijkl} \begin{cases} \hat{n}_i \\ \hat{n}_j \\ \hat{n}_k \\ \hat{n}_l \end{cases} = 0$

·) $\Lambda_{ijkl} \Lambda_{klmn} = \Lambda_{ijmn}$

REMARKS

1) If we didn't restrict to vacuum, we couldn't have chosen the TT gauge:

$$\Box \gamma_{\mu\nu} = -16\pi G\, T_{\mu\nu} \longrightarrow \Box \gamma = -16\pi G\, T$$

we have required $\tilde{\gamma} = 0$ but now: CONTRADICTION!

$$\tilde{\gamma} = \gamma - 2\partial_\rho \tilde{\xi}^\rho = 0 \longrightarrow \gamma = 2\partial_\rho \tilde{\xi}^\rho \longrightarrow \Box\gamma = 2\partial_\rho \Box \tilde{\xi}^\rho = 0$$

However, the polarisation states of a classical field are related to the spin of the massless particle that one expects upon quantisation of the theory. Therefore, the fact that GWs have two polarisation states is a representation of the fact that the graviton is a spin-2 massless field. This must be true in general, also if one isn't in vacuum and cannot exibihit the physical polarisation states by choosing the TT gauge. Indeed, it is possible to reduce the metric $h_{\mu\nu}$ to its physical degrees of freedom also without making the assumption of $T_{\mu\nu} = 0$ (Flanagan & Hughes gr-qc/0501041)

the linearised metric has:
1) gauge dofs
2) physical dof that are <u>not-radiative</u> and satisfy Poisson-like equations
3) GWs, that obey a wave equation and are the physical radiative dofs

To see this, one needs to construct <u>gauge-invariant variables</u>. Typical example: in cosmology, within the Friedmann metric, one is never in vacuum, one cannot go outside the "source".

2) How can one get the spin of the classical field?
Misner, Thorne, Wheeler chapter 35.6 give a heuristic relation between the spin of the field expected upon quantisation, and the angle θ under which the polarisation modes are invariant:

$$S = \frac{2\pi}{\theta}$$

(EM: 2 linear polarisation states, vectors return the same after rotation of 2π $\rightarrow$ $S = \frac{2\pi}{2\pi} = 1$

Let's perform a rotation of an angle θ around the z-axis of a GW propagating in the z-direction:

$$h'_{ij} = R_{ik} R_{jl} h_{kl}$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} h'_+ = h_+ \cos 2\theta - h_\times \sin 2\theta \\ h'_\times = h_+ \sin 2\theta + h_\times \cos 2\theta \end{cases}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_+\cos\theta - h_\times \sin\theta & h_\times\cos\theta + h_+\sin\theta & 0 \\ h_+\sin\theta + h_\times\cos\theta & h_\times\sin\theta - h_+\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} h_+\cos^2\theta - h_\times\cos\theta\sin\theta - h_+\sin^2\theta - h_\times\cos\theta\sin\theta & h_\times\cos^2\theta + h_+\cos\theta\sin\theta - h_\times\sin^2\theta + h_+\sin\theta\cos\theta & 0 \\ h_+\cos\theta\sin\theta - h_\times\sin^2\theta + h_+\sin\theta\cos\theta + h_\times\cos^2\theta & h_\times\sin\theta\cos\theta + h_+\sin^2\theta + h_\times\cos\theta\sin\theta - h_+\cos^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

invariant for $\theta = \pi$ $\begin{cases} h'_+ = h_+ \\ h'_\times = h_\times \end{cases}$ and therefore $S = \frac{2\pi}{\pi} = 2$

As we will see, if we decompose the radiation field into multipoles, all multipoles with $l < S$ go to zero, so sources with only moments $l < S$ cannot lead to that particular type of radiation.

EM: $s_\gamma = 1$, $\ell = 1$ is the first non-zero multipole of the radiation field, EM waves are generated by dipolar charges distributions (antenna) and cannot be generated by scalar charge distributions ($\ell = 0$)

GWs: $s_g = 2$, $\ell = 2$ is the first non-zero multipole of the radiation field (we will see this), GWs are generated by quadrupolar mass distributions and cannot be generated by scalar or dipole mass distributions.

(*)

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \qquad \left.\frac{d^2 x^i}{d\tau^2}\right|_{\tau=0} = \Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 \underset{\uparrow}{=} 0$$

$$\Gamma^i_{00} = 0$$

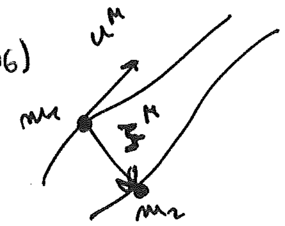
particles initially at rest remain at rest in the TT gauge

EFFECT OF GWs ON TEST MASSES:

Naively we expect GWs (to set) test masses into motion. This is a physical effect, which should then be invariant under coordinate transformation. However, the reference frame that we choose to describe it matters, as we will see. Detectors of GWs are constructed based on this physical effect, so it is important to model it properly.

The effect can be studied using the geodesic deviation equation, thinking about two masses freely falling on infinitesimally closed geodesics and studying the effect of the curvature due to GWs on this system.

$$u^\mu \nabla_\mu (u^\lambda \nabla_\lambda \xi^\nu) = - R^\nu{}_{\mu\lambda\sigma}\, \xi^\lambda u^\mu u^\sigma \quad (1.106)$$



Let's analyse this in the TT gauge since the metric is simple:

$$\begin{cases} g_{00} = -1 \\ g_{0i} = 0 \\ g_{ij} = \delta_{ij} + h^{TT}_{ij} \end{cases}$$

Christoffels: $\Gamma^0_{ij} = -\frac{1}{2}\partial_0 h^{TT}_{ij} \qquad \Gamma^i_{0j} = \frac{1}{2}\partial_0 h^{TT\,i}{}_j$

$$\Gamma^i_{jk} = \frac{1}{2}\left(\partial_j h^{TT\,i}{}_k + \partial_k h^{TT\,i}{}_j - \partial^i h^{TT}_{jk}\right)$$

The masses are initially at rest. The 4-velocity of the "observer" mass is:

$$\begin{cases} u^i = \dfrac{dx^i}{d\tau} \quad \text{with } u^i(\tau=0) = 0 \\ u^0 = \dfrac{dx^0}{d\tau} = \dfrac{1}{\sqrt{-g_{\mu\nu}\dfrac{dx^\mu}{dx^0}\dfrac{dx^\nu}{dx^0}}} = 1 \end{cases}$$

$\nu = i$ component of the equation becomes:

$$\nabla_0 (\nabla_0 \xi^i)\Big|_{\tau=0} = - R^i{}_{0j0}\, \xi^j \Big|_{\tau=0}$$

Riemann tensor in linearised theory:

$$R^\mu{}_{\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu \partial_\rho h^\mu{}_\sigma + \partial^\mu \partial_\sigma h_{\nu\rho} - \partial^\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h^\mu{}_\rho\right)$$

ATT !! The Riemann tensor in linearised theory is INVARIANT under coordinate transformation $x'^\mu = x^\mu - \xi^\mu$:

$$h^\mu{}_\sigma{}' = h^\mu{}_\sigma + \partial^\mu \xi_\sigma + \partial_\sigma \xi^\mu$$

$$R'^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + \frac{1}{2}\left(\partial_\nu \partial_\rho \partial^\mu \xi_\sigma + \cancel{\partial_\nu \partial_\rho \partial_\sigma \xi^\mu} + \right.$$
$$+ \cancel{\partial^\mu \partial_\sigma \partial_\nu \xi_\rho} + \partial^\mu \partial_\sigma \partial_\rho \xi_\nu$$
$$- \cancel{\partial^\mu \partial_\rho \partial_\nu \xi_\sigma} - \partial^\mu \partial_\rho \partial_\sigma \xi_\nu$$
$$\left. - \cancel{\partial_\nu \partial_\sigma \partial^\mu \xi_\rho} - \partial_\nu \partial_\sigma \partial_\rho \xi^\mu\right)$$

$$\boxed{R^i{}_{0j0} = -\frac{1}{2}\partial_0^2 h^{iTT}_j}$$ $\Rightarrow$ this is true in every coordinate system related to the TT frame with an infinitesimal coordinate transformation.

Going back to geodesic deviation: left hand side:

$$\nabla_0 (\partial_0 \xi^i + \Gamma^i_{0j}\xi^j) = \partial_0^2 \xi^i + (\partial_0 \Gamma^i_{0j})\xi^j + \Gamma^i_{0j}\,\partial_0 \xi^j + \Gamma^i_{0k}\,\partial_0 \xi^k +$$
$$+ \cancel{\Gamma^i_{0k}\Gamma^k_{0j}}\,\xi^j \quad \text{(second order in } h\text{)}$$

$\nabla_0 V^i = \partial_0 V^i + \Gamma^i_{0\rho} V^\rho$

$$= \partial_0^2 \xi^i + \underbrace{\frac{1}{2}\partial_0^2 h^i_j\, \xi^j}_{\text{cancels the Riemann term}} + \partial_0 h^i_j\, \partial_0 \xi^j$$

(omit TT for simplicity)

$$\partial_0^2 \xi^i \Big|_{\tau=0} = -\left[\partial_0 h^i_j \cdot \partial_0 \xi^j\right]_{\tau=0}$$

with the geodesic
$$\frac{d^2x^i}{d\tau^2}\Big|_{\tau=0} = \left[-\Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\right]_{\tau=0}$$
(*)

so if at time $\tau=0$ the (time derivative of) infinitesimal displacement between the two geodesics is zero, it remains zero also at $\tau>0$!
Masses do not move under the influence of GWs in the TT gauge. This is an artefact of the choice of coordinates which illustrates well the misinterpretations that can occur because of gauge effects. It does not mean that the passing GW has no physical effect on test masses, but that in the TT frame the coordinates stretch themselves as the GW passes though (it is as if the coordinates are marked by a network of freely-falling masses).

Indeed, if one looks at the PROPER DISTANCE between the two neighbouring masses, one should pick up the physical effect!

$$m_1 : \ (t, x_1, 0, 0)$$
$$m_2 : \ (t, x_2, 0, 0)$$

coordinate distance $\xi = x_1 - x_2$ remains constant.

Proper distance:

$$ds^2 = \cancel{-dt^2} + \cancel{dz^2} + [1 + h_+ \cos(\omega(t-z))]\,dx^2 + \cancel{[1 - h_+ \cos(\omega(t-z))]\,dy^2}$$
$$\cancel{+ 2 h_\times \cos(\omega(t-z))\,dx\,dy}$$

$$s = \sqrt{1 + h_+ \cos(\omega(t-z))}\ \xi \simeq \xi\left(1 + \frac{1}{2} h_+ \cos(\omega(t-z))\right)$$

The proper distance changes as the GW passes by in the TT gauge!

CAN WE CONSTRUCT A DEVICE WHICH MONITORS "PROPER DISTANCES" BETWEEN NEIGHBOUR MASSES?

In a <u>local inertial frame</u> proper distance $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ is the same as coordinate distance $\sqrt{dx_1^2 + dx_2^2 + dx_3^2}$ up to first order in curvature effects:

By the equivalence principle, it exists a frame for which

<u>local inertial frame</u>

!! no more TT gauge

$$\begin{cases} g_{\mu\nu}(P) = \eta_{\mu\nu} \\ \partial_\lambda g_{\mu\nu}(P) = 0 \Leftrightarrow \Gamma^\lambda_\mu(P) = 0 \end{cases}$$

in the neighborhood of P, the metric can be Taylor expanded:

$$g_{\mu\nu} = \underbrace{g_{\mu\nu}(P)}_{\eta_{\mu\nu}} + \underbrace{\tfrac{1}{2}\partial_\alpha \partial_\beta g_{\mu\nu}}(x^\alpha - x^\alpha_P)(x^\beta - x^\beta_P) + O(x - x_P)^3$$

standard Taylor expansion of a function $\mathbb{R}^n \to \mathbb{R}$

curvature effects (like GWs) remain: these cannot be eliminated by the choice of a reference frame!

take result of Ni & Zimmermann PRD 17, 6 (1978) (see also Manasse & Misner J. Math. Phys. 735 (1963))

(differs from Sharoman chp. 3.2.2 by some factors, and also calculates time part) TO BE VERIFIED

$$ds^2 = -dt^2\,[1 + R_{0\ell 0m}\, x^\ell x^m] + 2\,dx^i dt\,\left(\tfrac{2}{3} R_{0\ell im}\, x^\ell x^m\right) + dx^i dx^j \left(\delta_{ij} - \tfrac{1}{3} R_{i\ell jm}\, x^\ell x^m\right)$$

corrections to the flat metric arise at order $\left(\frac{L}{L_B}\right)^2$

<u>Can we construct a detector on Earth such that it coincides with a local inertial frame?</u> the answer is yes if one adopts some tricks

on Earth we are accelerated observers : our metric is :

$$ds^2 \simeq -dt^2\left[1 + 2\underbrace{\underline{a}\cdot\underline{x}}_{\text{gravitational acc.}} + (\underline{a}\cdot\underline{x})^2 - (\underbrace{\underline{\Omega}\times\underline{x}}_{\text{Earth rotation}})^2 + R_{0k0e}x^k x^e\right]$$
$$+ 2\,dt\,dx^i\left[\underline{\Omega}\times\underline{x} - \frac{2}{3}R_{0eim}x^e x^m\right]$$
$$+ dx^i dx^j\left[\delta_{ij} - \frac{1}{3}R_{iejm}x^e x^m\right]$$

how to get to this?

1) detectors have very refined suspension mechanisms which cancel the effect of acceleration and rotation, and effectively put the masses in free fall in the (x,y) plane

2) they still need to distinguish GWs from the effect of time-varying gravitational potential (such as seismic noise). these are eliminated by choosing a proper frequency window of operation :

$$\underbrace{0.1\,Hz}_{\text{the lowest frequency possible, below has much seismic noise}} < f < \underbrace{10^3\,Hz}_{\text{above this frequency, other noises like laser noise, thermal...}} \implies$$

in this window the detector is relatively isolated from noises due to time-varying Earth grav. field, and there are sources in this window!

Thanks to the suspensions and to the operation frequency, a detector can be described in the effective metric which is the one of a freely-falling observer, in which one remains ONLY the Riemann tensor terms!

what is the geodesic deviation equation in the detector frame?

$$\begin{cases} u^i = \dfrac{dx^i}{d\tau} \quad \text{with} \quad u^i(\tau=0) = 0 \\ u^0 = \dfrac{dx^0}{d\tau} = \dfrac{1}{\sqrt{1-\left(\dfrac{dx^i}{dx^0}\right)^2}} \simeq 1 \end{cases}$$

neglect terms of order h

since the detector does not have relativistic velocity

for masses initially at rest:

$$\partial_0^2 \xi^i + \underbrace{(\partial_0 \Gamma^i_{0j})\, \xi^j}_{} = -R^i{}_{0j0}\, \xi^j$$

this can be written in the TT gauge because it is INVARIANT

this is also 0 as the metric only depends on x^i (spatial components)

christoffel symbols are zero for local inertial observer

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{i\,TT}_{j}\, \xi^j}$$

NB! this applies for "infinitesimal" distance between the geodesics, with respect to the typical scale over which the curvature (GWs) vary:

$$\underset{\text{detector size}}{L} \ll \underset{\text{GW wavelength}}{\lambda}$$

Now we can finally study the effect of GWs on a ring of test masses, using the above equation which is the one suited to our observer measuring the coordinate (proper) distance between them:

Suppose wave in the z direction; masses in the plane $z=0$;

coordinate distance : $\xi^i(z=0,t) = (x_0 + \delta x(t)\,,\, y_0 + \delta y(t)\,,\, 0)$

x_0: initial mean position; $\delta x(t)$: time-dependent, wave-induced displacement

For the + polarization, the GW is : $h_{ij}^{(+)} = h_+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin\omega t$

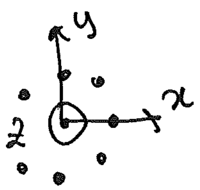
$$\ddot{h}_{ij}^{(+)} = h_+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (-\omega^2) \sin\omega t$$

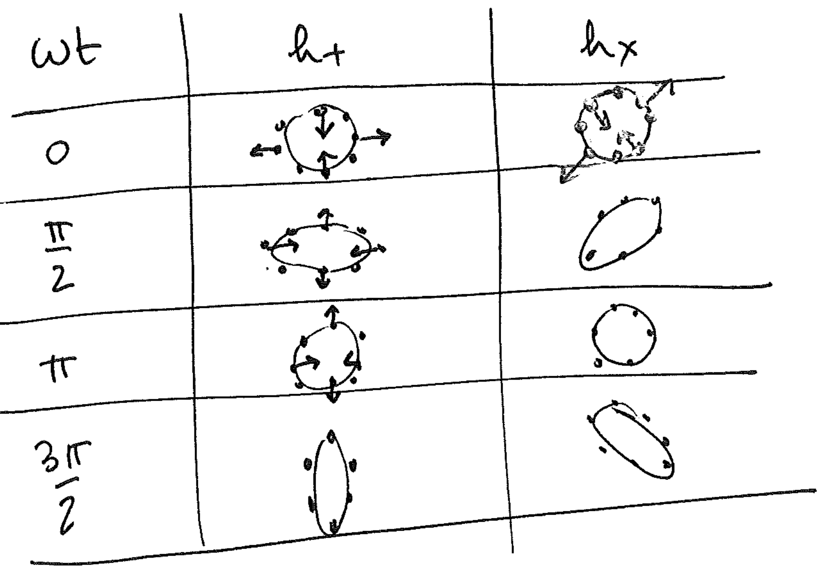
$$\begin{cases} \delta\ddot{x}(t) = \frac{1}{2} h_+ (-\omega^2) \sin\omega t \,(x_0 + \cancel{\delta x(t)}) \\ \delta\ddot{y}(t) = \frac{1}{2} h_+ \omega^2 \sin\omega t \,(y_0 + \underbrace{\cancel{\delta y(t)}}_{\text{neglected since second order in } h!}) \end{cases}$$

$$\begin{cases} \delta x(t) = \frac{h_+}{2} x_0 \sin\omega t \\ \delta y(t) = -\frac{h_+}{2} y_0 \sin\omega t \end{cases}$$

and similarly for the $h_\times$ polarization

$$\begin{cases} \delta x(t) = \frac{h_\times}{2} y_0 \sin\omega t \\ \delta y(t) = \frac{h_\times}{2} x_0 \sin\omega t \end{cases}$$





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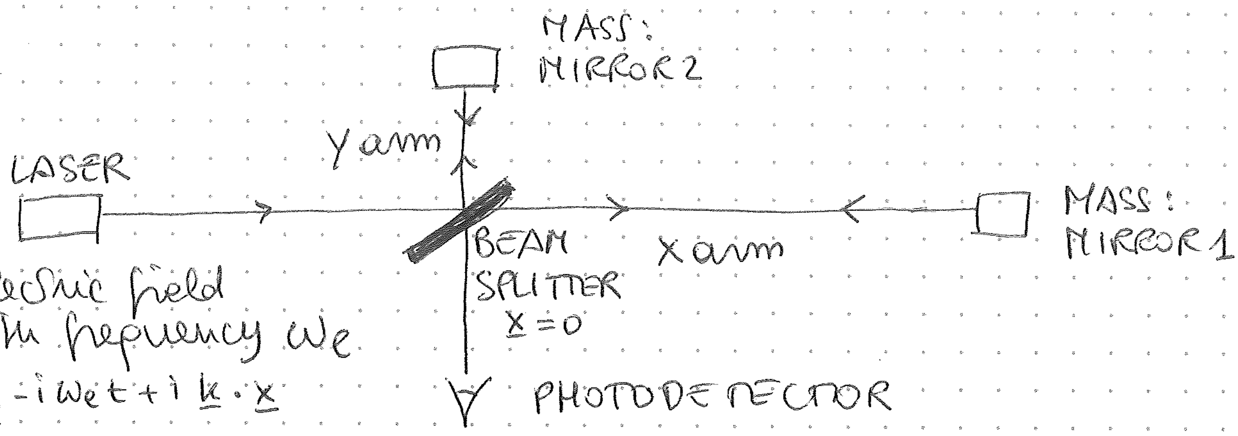
PRINCIPLES OF GW DETECTION WITH INTERFEROMETERS



Electric field with frequency ω_e

$E_0 e^{-i\omega_e t + i \underline{k} \cdot \underline{x}}$

The instrument measures changes in the travel time of the laser beam in the two arms: at the beam splitter position $\underline{x} = 0$ and at time t we have the superposition of a beam that entered the beam splitter at times:

beam that went through x arm entered at $t_0^x = t - 2L_x$

beam that went through y arm entered at $t_0^y = t - 2L_y$

Since the phase (i.e. the value of the exponent in the electric field) of the laser is constant during propagation, the electric fields that combine at the beam splitter are:

$$\begin{cases} E_1 = -\frac{1}{2} E_0 \, e^{-i\omega_e (t - 2L_x)} & \text{initial phase, when the photon entered the x-arm} \\ E_2 = \frac{1}{2} E_0 \, e^{-i\omega_e (t - 2L_y)} & \text{initial phase when the photon entered the y-arm} \end{cases}$$

$\rightarrow$ coefficients due to the reflection

The output is:

$$E_1 + E_2 = \frac{1}{2} E_0 \, e^{-i\omega_e t + i\omega_e (L_x + L_y)} \Big[-e^{\overbrace{2i\omega_e L_x - i\omega_e (L_x + L_y)}^{i\omega_e (L_x - L_y)}} + e^{\underbrace{2i\omega_e L_y - i\omega_e (L_x + L_y)}_{-i\omega_e (L_x - L_y)}} \Big] =$$

$$= i E_0 \, e^{-i\omega_e t + i\omega_e (L_x + L_y)} \sin [\omega_e (L_x - L_y)]$$

So the power measured at the observer is

$$|E_1+E_2|^2 = E_0^2 \sin^2[w_e(L_x-L_y)]$$

So the power of the photodetector is modulated by the difference in the lenght between the arms. This is a physical effect that must be the same regardless of the reference frame. In the detector frame it is clear, since the GWs passing by displace the mirrors at the end of the arms (they are massive) from their original position and change therefore the lenght of the arms causing the varying power output. If we choose to describe the detector in the TT gauge, we must find the same physical effect: but in this gauge, the mirrors (masses) are attached to the coordinates and their coordinate distance remains the same. How does it work? The effect of the GW manifests itself by affecting the PROPAGATION OF THE LIGHT between the mirrors.

Let us fix the TT gauge. Suppose a GW with only + polarisation and coming from the z-direction perpendicular to the detector

$$h_+(t) = h_0 \cos(w_{GW} t)$$

$$ds^2 = -dt^2 + [1+h_+(t)]dx^2 + [1-h_+(t)]dy^2 + dz^2$$

the photons of the laser travel on null geodesics, and therefore the light in the x-arm satisfies:

$$ds^2 = 0 \quad \rightarrow \quad dx \simeq \pm dt \left[1 - \frac{h_+(t)}{2}\right] \quad \begin{pmatrix}\text{Taylor expansion} \\ \text{in } h_+\end{pmatrix}$$

$$\underbrace{\int_0^{L_x} dx - \int_{L_x}^{0} dx}_{\text{full photon path}} = \underbrace{\int_{t_0^x}^{t_1} dt' \left[1 - \frac{h_+(t')}{2}\right]}_{\text{going}} - \underbrace{\int_{t_1}^{t} -dt' \left[1 - \frac{h_+(t')}{2}\right]}_{\substack{\text{coming back} \\ \text{(negative sign)}}}$$

$$2L_x = \int_{t_0^x}^{t} dt' \left[1 - \frac{h_+(t')}{2}\right] = (t - t_0^x) - \frac{1}{2}\int_{t_0^x}^{t} dt'\, h_+(t')$$

the total time interval that one photon spends in the x-arm is therefore

$$(t - t_0^x) \simeq 2L_x + \frac{1}{2}\int_{t_0^x}^{t_0^x + 2L_x} dt'\, h_0 \cos(\omega_{GW} t')$$

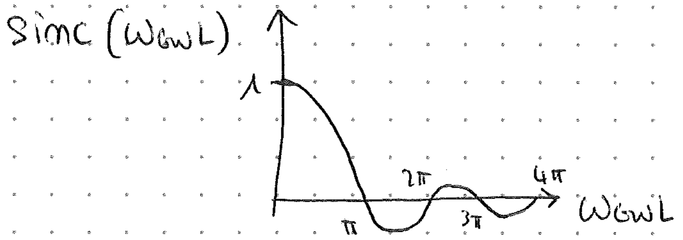
$\to$ here we neglect the effect of the GW, which would give a second order term

$$\simeq 2L_x + \frac{1}{2\omega_{GW}} h_0 \left[\sin(\omega_{GW}(t_0^x + 2L_x)) - \sin(\omega_{GW} t_0^x)\right] \qquad t = t_0^x + 2L_x$$

$$\simeq 2L_x + \frac{h_0}{2\omega_{GW}} \left[2\sin(\omega_{GW} L_x)\cos(\omega_{GW}(t_0^x + L_x))\right]$$

$$\simeq 2L_x + L_x \frac{\sin(\omega_{GW} L_x)}{\omega_{GW} L_x} h_+(t_0^x + L_x)$$

So there is a correction in the time the photon spent in the x-arm with respect to flat spacetime due to the presence of the GW. The function that determines the amplitude of the correction, on top of h_+ is such that:



- if $\omega_{GW} L \ll 1$ the time shift is just h_+
- if $\omega_{GW} L \gg 1$ the effect cancels out since the GW does many oscillations.

One can now calculate the output in the detector accounting for the fact that the phase does not change. For this one needs to express t_0^x and t_0^y:

$$\begin{cases} t_0^x \simeq t - 2L_x - L_x \,\text{sinc}(\omega_{GW} L_x)\, \underbrace{h_+(t-L_x)}_{} \\ t_0^y = t - 2L_y + L_y \,\text{sinc}(\omega_{GW} L_y)\, h_+(t-L_y) \end{cases}$$

sign difference coming from the one of the metric

neglect the effect of GWs: $t_0^x + L_x \simeq t - 2L_x + L_x$

so the total electric field at the beam splitter at time t becomes:

$$E_1 + E_2 = -\frac{1}{2}E_0\left(e^{-i\omega_e t_0^x} - e^{-i\omega_e t_0^y}\right) = -\frac{1}{2}E_0\, e^{-i\omega_e(t-2L)}$$

$$\left[e^{-i\omega_e(-2L_x - L_x \text{sinc}()h_+ + 2L)} - e^{-i\omega_e(-2L_y + L_y \text{sinc}()h_+ + 2L)}\right]$$

with $L = \frac{L_x+L_y}{2}$ and therefore $\begin{cases} 2L_x = 2L + (L_x - L_y) \\ 2L_y = 2L - (L_x - L_y) \end{cases}$

$$= -\frac{1}{2}E_0\, e^{-i\omega_e(t-2L)}\left[e^{-i\omega_e(L_y - L_x - L_x \text{sinc}()h_+)} - e^{-i\omega_e(L_x - L_y + L_y \text{sinc}()h_+)}\right]$$

$$= -\frac{1}{2}E_0\, e^{-i\omega_e(t-2L)}\left[e^{i\omega_e(L_x - L_y + L\,\text{sinc}()h_+)} - e^{-i\omega_e(L_x - L_y + L\,\text{sinc}()h_+)}\right]$$

term prop to h_+ so set here $L_x \simeq L_y \simeq L$

$$= -iE_0\, e^{-i\omega_e(t-2L)} \sin\left[\omega_e(L_x - L_y) + \omega_e L\, \text{sinc}(\omega_{GW} L)\, h_+(t-L)\right]$$

$$= -iE_0\, e^{-i\omega_e(t-2L)} \sin\left[\phi_0 + \Delta\phi\right]$$

a phase that is not relevant, it can be adjusted experimentally

Effect due to the GWs is a dephasing

$$\boxed{\Delta\phi = \omega_e L\ \text{sinc}(\omega_{GW} L)\, h_+(t-L)}$$

In the case without GWs the effect due to the difference in the arm lenght was $\omega_e \Delta L$ so that correspondingly here we have an equivalent arm lenght difference such that

$$\frac{\Delta L}{L} = \mathrm{sinc}(\omega_{GW} L)\, h_+(t-L)$$

In order to MAXIMIZE the dephasing effect, one must choose the parameters of the instrument such that

$$\mathrm{Max}[\Delta\phi] = \mathrm{Max}\left[\sin(L\omega_{GW})\frac{\omega_e}{\omega_{GW}}\right] \Rightarrow L = \frac{\pi}{2}\frac{1}{\omega_{GW}}$$

optimal lenght of the arms

A frequency of the GWs that is rich of signals and still manageable on Earth is $f \simeq 100\ Hz$, because it is in the middle of the range between the seismic noise effects and the laser noise. This means: $\left(f = \frac{\omega c}{2\pi}\right)$

$$L = \frac{\pi}{2}\,\frac{c}{2\pi \cdot 100\,Hz}\left(\frac{100\,Hz}{f}\right) = \frac{1}{400}\, c\ km \left(\frac{100\,Hz}{f}\right)$$

restores the right dimensions

$$\simeq 750\ km \left(\frac{100\,Hz}{f}\right)$$

This is clearly not possible on Earth, it is too long. Indeed the Earth-based interferometers have arms of about 3/4 km. They use a system called Fabry-Pérot cavities that effectively fold the 750 km arm into 3/4 km by reflecting the laser many times back and forth. The effective lenght becomes a factor $\frac{2F}{\pi}$ times larger, where F is called the "finesse" of the cavity.

$$L_{eff} \simeq \frac{\pi}{2F}\underbrace{750\ km}_{L} \simeq 4\ km \quad \text{for } F \simeq 300$$

The dephasing one needs to measure is:

$$\Delta\phi \simeq \omega_e \frac{2F}{\pi} L_{eff} h_+ \simeq \frac{1}{10^{-10}\,\text{km}} \frac{600}{\pi} 4\,\text{km}\, h_+ \simeq 10^{13} h_+$$

we assume $\omega_{GW} L \ll 1$ that is always the case for Earth-based interferometers

$f \simeq 10^2 - 10^3$ Hz $\rightarrow \frac{\lambda}{2\pi} \simeq 500 - 50$ km $\gg L \simeq 4$ km

set $F = 300$ and $\omega_e = \frac{1}{\lambda_e} = (0.1\,\mu\text{m})^{-1}$ and $L_{eff} = 4$ km

a typical value of h_+ as we will see at the end of the course is $h_+ \simeq 10^{-21}$ so the total dephasing is of the order of $\Delta\phi \simeq 10^{-8}$ rad

$\boxed{\text{THE ENERGY MOMENTUM TENSOR OF GWs}}$

From the fact that GWs displace test masses it is clear that they carry energy and momentum. What is the expression for the energy-momentum tensor of GWs?
According to GR, every form of energy and momentum contributes to the curvature of space time: so the question is equivalent to: are GWs a source of space-time curvature?
The answer to this question will allow us to find an expression for the GW energy-momentum tensor.
However, to answer this question one must go beyond linearized theory in flat space time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

1. This is defined as GWs because it is the part of the metric that satisfies a wave eq. in the Hilbert gauge

2. However, keeping this as Minkowski one excludes since the beginning the possibility that GWs can create any curvature in the bck space-time

So, the new expansion is:

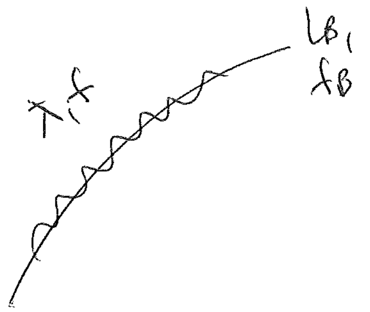
$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\substack{\text{curved dynamical bck metric} \\ \text{accounting for the effect of GWs}}} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

However, in this setting, it is non-trivial to distinguish

The background from the perturbation, because the bck is no longer Minkowski. The way to proceed is to exploit the fact that there is a neat separation of scales / frequencies.

Let us suppose the following conditions:

$\bar{g}_{\mu\nu}$	$+$	$h_{\mu\nu}$
the bck metric has a typical scale of spatial variation / or temporal variations		the small amplitude perturbation has a typical scale of spatial variation / temporal variation



$L_B \gg ƛ \rightarrow$ $h_{\mu\nu}$ are small ripples on a smoother bck

$f_B \ll f \rightarrow$ $h_{\mu\nu}$ are a fast evolving perturbation on a slowly evolving bck.

why $ƛ$? suppose $h_{\mu\nu}$ plane wave $e^{i \underline{k} \cdot \underline{x}}$ with $k = \frac{2\pi}{\lambda}$ so lenght $\underline{x}$ to be compared with reduced wave lenght $ƛ = \frac{\lambda}{2\pi}$

The effective method to perform the distinction among the GWs perturbations and the bck, based on their typical wavelenght / frequency is to PERFORM AN AVERAGE

-) Volume average over a region $\bar{\ell}^3$ with $ƛ \ll \bar{\ell} \ll L_B$

  $\langle \dots \rangle_{\bar{\ell}}$

-) time average over time $\bar{t} = 1/\bar{f}$ with $f_B \ll \bar{f} \ll f$

  $\langle \dots \rangle_{\bar{f}}$

By performing this average, one can find the energy momentum tensor of GWs. First one needs to write down Einstein's equations up to II order in $h_{\mu\nu}$, and then average them. This is a highly non-trivial procedure that can be found e.g. in

- M. Maggiore "GWs" Volume 1 2008
- Misner, Thorne, Wheeler "Gravitation" 1997

The averaged - background Einstein's eqs are:

$$\underbrace{\langle R_{\mu\nu}\rangle_{\bar{\ell},\bar{t}} - \frac{1}{2}\bar{g}_{\mu\nu}\langle R\rangle}_{\substack{\text{The dynamics of the}\\ \text{bck. spacetime, i.e.}\\ \text{the bck metric } \bar{g}_{\mu\nu}\\ \text{is determined by:}}} = 8\pi G\,(\underbrace{\langle T_{\mu\nu}\rangle}_{1.} + \underbrace{t_{\mu\nu}}_{2.})$$

1. $\langle T_{\mu\nu}\rangle$ is the background energy mom. tensor of matter it can be there or not, it is completely independent on the GWs, but it influences $\bar{g}_{\mu\nu}$
2. $t_{\mu\nu}$ denotes the $\boxed{\text{ENERGY MOM. TENSOR OF THE GWS}}$

$$\boxed{t_{\mu\nu} = \frac{1}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\,\partial_\nu h^{\alpha\beta}\rangle_{\bar{\ell},\bar{t}}}$$

Important: as previously hinted to, this quantity is quadratic in $h_{\mu\nu}$ (second order): this is the reason why it can influence the background.

From a term that is quadratic in the perturbation, the average selects the long wavelenght / small frequency component:

- first order in $h_{\mu\nu}$: average goes to zero $\langle \sin x \rangle = 0$
- second order in $h_{\mu\nu}$: the average does not go to zero, as two short-wavelength modes can combine into a long-wave lenght mode: $\langle \sin^2 x \rangle \neq 0$

  two modes $\underline{k}_1 \simeq -\underline{k}_2$ combine to give $\underline{k} \simeq 0$ a long-wave lenght mode

Another example is the F.T. of a quadratic function

$$FT[h^2(x)] = \int_0^\infty dk'\, h(k')\, h(k-k') = H(k) \qquad \lim_{k\to 0} H(k) = \int_0^\infty dk'\, h^2(k') \neq 0$$

$\uparrow$ convolution

$H(k)$ has contributions to long wavelenght $k \to 0$

The $\boxed{\text{GW ENERGY DENSITY}}$ in the TT gauge becomes:

$$\rho_{GW} = t^{00} = \frac{1}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}^{ij\,TT} \rangle_{\bar{\ell}, \bar{t}}$$

This quantity can only be defined whithin a volume or time average, necessary to capture the effect of the GWs on the background. Another way to under-stand the average is the equivalence principle: one can always find coordinates in which space-time is flat in a point P. So one can always gauge away the presence of any curvature (including GWs) in a point P. Therefore, GWs energy density needs

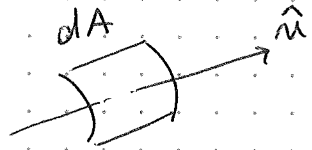
to be defined within a given volume / time interval.
By decomposing the GW in plus and cross polarizations one obtains:

$$h_{ij}^{TT} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \qquad \left(\begin{array}{l} e_{ij}^+ e_{ij}^+ = 2 \\ e_{ij}^\times e_{ij}^\times = 2 \\ e_{ij}^+ e_{ij}^\times = 0 \end{array} \right)$$

$$\dot{h}_{ij}^{TT} \dot{h}^{TT\,ij} = 2 \left(\dot{h}_+^2 + \dot{h}_\times^2 \right)$$

$$\rho_{GW} = \frac{1}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

The $\boxed{\text{GW energy flux}}$ is the energy rate going through a surphace:



$$\frac{dE_{GW}}{dt\, dA} = \hat{n}_e \underbrace{t^{0e}}_{\text{energy flux}} = -\hat{n}_e t_{0e} = -\frac{\hat{n}_e}{32\pi G} \langle \dot{h}_{ij}^{TT} \partial_e h^{ij\,TT} \rangle$$

consider now that the GW is a plane wave propagating in the direction $\hat{k} \parallel \hat{n}$; $h_{ij}^{TT}(\underline{x} \cdot \hat{n} - t)$

$$\hat{n}_e \partial_e h^{ij\,TT}(\underline{x} \cdot \hat{n} - t) = \hat{n}_e \frac{\partial h^{ij\,TT}(\underline{x} \cdot \hat{n} - t)}{\partial(\underline{x} \cdot \hat{n} - t)} \frac{\partial(\underline{x} \cdot \hat{n} - t)}{\partial x^e} =$$

$$= \underbrace{\hat{n}_e \hat{n}^e}_{1} \frac{\partial h^{ij\,TT}(\underline{x} \cdot \hat{n} - t)}{\partial t} \frac{\partial t}{\partial(\underline{x} \cdot \hat{n} - t)} = -\frac{\partial h^{ij\,TT}}{\partial t}$$

So the GW energy flux becomes:

$$\frac{dE_{GW}}{dt\, dA} = \frac{1}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}^{ij\,TT} \rangle = \frac{1}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

And therefore the POWER PER UNIT SOLID ANGLE:

$$\frac{dP}{d\Omega} = \frac{dE_{GW}}{dt\, dA} r^2 = \frac{r^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$ (useful for the quadrupole formula)

$dA = r^2 d\Omega$ r is the distance of the surphace from the source of GWs

GENERATION OF GWs IN LINEARISED THEORY

We study the generation of GWs maintaining the linearisation of the metric around Minkowski (linearised theory). This means that the space-time around the source isn't deformed by the gravitational field of the source itself. Therefore, the gravitational field generated by the source must be WEAK and the source can be described by Newtonian theory.

For a self-gravitating system, weak gravitational field means that the velocity is small:

two-body system held together by the gravitational force:

m_1 —— r —— m_2

$m = m_1 + m_2$ total mass

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ reduced mass

in the com frame, equivalent to one particle of mass μ in gravitational field of m.

$$E_{kin} = -\frac{1}{2} U$$

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{G\mu m}{r}$$

$$v^2 = \frac{Gm}{r} = \phi \quad \text{Newtonian potential}$$

$$\boxed{\phi \ll 1 \leftrightarrow v \ll 1}$$

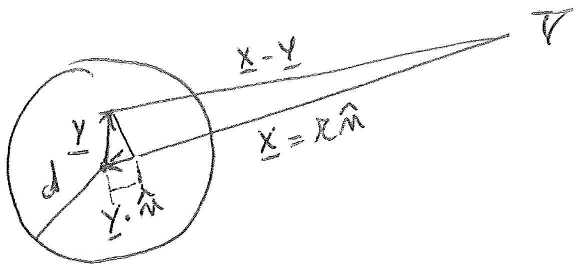
when solving for the GW emission, we will then perform an expansion in the small parameter

$v \ll 1$

(actually to be understood as $\frac{v}{c} \ll 1$)

Let us consider the generic retarded solution in eq. (2.30) for the weak gravitational field in the Hilbert gauge:

$$\bar{\gamma}_{\mu\nu} = 4G \int \frac{T_{\mu\nu}(x^0 - |\underline{x}-\underline{y}|, \underline{y})}{|\underline{x}-\underline{y}|} d^3y$$



We take the point of view of an observer far away from the GW source. We can then PROJECT ON THE TT GAUGE using the projector defined in page 11, $\Lambda_{ijkl}(\hat{n})$

only the space components are relevant

$$h_{ij}^{TT}(\underline{x},t) = 4G\, \Lambda_{ijkl}(\hat{n}) \int \frac{T^{kl}(t-|\underline{x}-\underline{y}|, \underline{y})}{|\underline{x}-\underline{y}|} d^3y$$

because the observer is far-away from the GW source, one can simplify:

$$|\underline{x}-\underline{y}| = \sqrt{r^2 - 2r\,\underline{y}\cdot\hat{n} + |\underline{y}|^2}$$

$$= r\sqrt{1 - 2\frac{\underline{y}\cdot\hat{n}}{r} + \frac{|\underline{y}|^2}{r^2}}$$

$$\simeq r\left(1 - \frac{\underline{y}\cdot\hat{n}}{r} + \mathcal{O}\left(\frac{d}{r}\right)^2\right)$$

$$= r - \underline{y}\cdot\hat{n} + \mathcal{O}\left(\frac{d^2}{r}\right)$$

where $|\underline{y}| \leq d$, d the size of the source

neglect terms of $\mathcal{O}(\frac{d}{r})$ in the amplitude and $\mathcal{O}(\frac{d^2}{r})$ in the integral

$$h_{ij}^{TT}(\underline{x},t) \simeq \frac{4G}{r} \Lambda_{ijkl}(\hat{n}) \int T^{kl}(t - r + \underline{y}\cdot\hat{n}, \underline{y})\, d^3y$$

$t-r$ is the time it takes to the signal to go from the centre of the source to the observer at $\underline{x} = r\hat{n}$

$\underline{y}\cdot\hat{n}$ is the time it takes to the signal to go through the source. Since $d \ll r$, we can consider this as a

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small parameter on which we can expand:

$$T^{ke}(t-r+\underline{y}\cdot\hat{n},\underline{y}) \simeq T^{ke}(t-r,\underline{y}) + \underline{y}\cdot\hat{n}\,\partial_0 T^{ke}\Big|_{t-r} + \\ + (\underline{y}\cdot\hat{n})^2\,\partial_0^2 T^{ke}\Big|_{t-r} + \mathcal{O}(\underline{y}\cdot\hat{n})^3$$

Let us now demonstrate that this expansion is in reality equivalent to a low-velocity expansion.

Suppose the source has a characteristic velocity v.
Then, $d \sim v\, t_s$ where t_s is the typical time of evolution of the source. Therefore: $\partial_0 T^{ke} \sim \dfrac{T^{ke}}{t_s}$
so that the first term in the expansion is:

$$\underline{y}\cdot\hat{n}\,\partial_0 T^{ke} \sim \hat{\underline{y}}\cdot\hat{n}\,\frac{T^{ke}}{t_s} \leq d\,\frac{T^{ke}}{t_s} \sim v\,T^{ke} \underset{\boxed{v \ll 1}}{\ll} T^{ke}$$

Neglecting all terms in the expansion, one finally writes:

$$h_{ij}^{TT}(\underline{x},t) \simeq \frac{4G}{r}\,\Lambda_{ijke}(\hat{n}) \int d^3y\; T^{ke}(t-r,\underline{y}) \qquad (1)$$

at zeroth order in the low-velocity expansion.
The integral can be expressed in terms of the moments of the mass distribution:

$\rho = T^{00}$ is the energy density of the source

0) $M = \int d^3y\, \rho(\underline{y})$ zeroth moment, the TOTAL MASS

1) $M^i = \int d^3y\, \rho(\underline{y})\, y^i$ first moment, the CENTRE OF MASS (vector $\dfrac{M^i}{M}$)

2) $M^{ij} = \int d^3y\ \rho(\vec{y})\ y^i y^j$ second moment, the TENSOR MOMENT OF INERTIA

The second moment can be related to the integral in eq.(1.1) by using the conservation of energy and momentum of the source $\partial_\mu T^{\mu\nu} = 0$

as we have seen, this is valid in the Hilbert gauge and it is satisfied in linearised theory because the back-reaction of the gravitational potential on the source is neglected. The source is described in Newtonian theory and the bck. spacetime is Minkowski. This also means that the source does not loose energy by the GW emission.

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = 0$$

$$\ddot{M}^{ij} = \left(\int d^3y\ \dot{T}^{00}\ y^i y^j\right)^{\cdot} = \left(-\int d^3y\ \partial_k T^{k0}\ y^i y^j\right)^{\cdot} =$$

$$= \Big(\underbrace{-\int d^3y\ \partial_k (T^{0k} y^i y^j)} + \int d^3y\ T^{0k}\ \delta^i_k\, y^j + \int d^3y\ T^{0k}\ y^i\, \delta^j_k\Big)^{\cdot}$$

total derivative, equal to the integral over the surface by STOKES' theorem, can be put to zero

$$= \int d^3y\ \dot{T}^{0i}\ y^j + \int d^3y\ \dot{T}^{0j}\ y^i$$

$$= -\int d^3y\ \partial_k T^{ki}\ y^j - \int d^3y\ \partial_k T^{kj}\ y^i$$

$$\partial_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_j T^{ij} = 0$$

$$= -\int d^3y\ \Big[\underbrace{\partial_k (T^{ki} y^j + T^{kj} y^i)}_{\text{total derivative}}\Big] + \int d^3y\ \left[T^{ki}\,\delta^j_k + T^{kj}\,\delta^i_k\right]$$

and so in the end one has $\ddot{M}^{ij} = 2\int d^3y \, T^{ij}$ and therefore:

$$h_{ij}^{TT}(x,t) \simeq \frac{2G}{r} \Lambda_{ij\,kl}(\hat{n}) \, \ddot{M}^{kl}(t-r)$$

From this expression one can already appreciate that a STATIC SOURCE CANNOT EMIT GWs. We can still decompose the moment of inertia into a pure trace and a traceless part: the latter is the only one surviving from the TT projection:

$$\Lambda_{ij\,kl}(\hat{n}) \, \ddot{M}^{kl} = \Lambda_{ij\,kl}(\hat{n}) \left(M^{kl} - \frac{1}{3}\delta^{kl} M \right)^{\cdot\cdot}$$

we subtract the trace that doesn't count

$$= \Lambda_{ij\,kl}(\hat{n}) \left[\int d^3y \, \rho(y) \left(y^k y^l - \frac{1}{3}\delta^{kl} y^2 \right) \right]^{\cdot\cdot}$$

$$= \Lambda_{ij\,kl}(\hat{n}) \, \ddot{Q}^{kl}$$

where Q^{kl} is the QUADRUPOLE MOMENT OF THE MASS DENSITY

finally we have arrived at:

$$\boxed{h_{ij}^{TT}(x,t) \simeq \frac{2G}{r} \Lambda_{ij\,kl}(\hat{n}) \, \ddot{Q}^{kl}(t-r)} \qquad (2)$$

GWs are radiated by NON-STATIC MASS DISTRIBUTIONS THAT POSSESS AT LEAST A QUADRUPOLE MOMENT. Monopole and dipole radiation are absent for GWs: Indeed, as we have seen in the Nearly Newtonian chapter and in the chapter about the Lense-Thirring effect, the monopole and the dipole are static components of the gravitational field and do not radiate:

Eq. (2.32) – (2.33) : $\gamma_{00} = -4\phi = -4G \int d^3y \, \frac{T_{00}(t,y)}{|x-y|}$

Eq. (2.48) : $\gamma_{0i} = 4A_i(x) = 4G \int d^3y \, \frac{T_{0i}(t,y)}{|x-y|}$

One could interpret this fact within a Newtonian viewpoint:

the monopole is $M = \int d^3y \, T^{00}(y)$ the total mass

the dipole is $P^i = \int d^3y \, T^{0i}(y)$ the momentum

these quantities are conserved in the absence of external forces, $\dot{M} = \dot{P}^i = 0$

$\downarrow$ $\hookrightarrow \int d^3y \, \dot{T}^{0i} = -\int d^3 \partial_j T^{ij}$ total derivative

$\int d^3y \, \partial_0 T^{00} = -\int d^3y \, \partial_i T^{0i}$ total derivative

and since a static distribution of mass does not radiate GWs, then those quantities do not radiate GWs.

However, this interpretation is not correct. The fact that the total mass and momentum are conserved is only true strictly in linearized theory over Minkowski. But in reality a GW source does loose energy and momentum by radiating GWs. So M and P^i are not conserved.

One can change approach and use a Post-Newtonian expansion that fully accounts for the back-reaction of the GW emission of the source, and still one finds that the monopole and dipole, even if not static, do not emit GWs. Indeed, as discussed previously, the fact that GWs are sourced at minimum by a quadrupole is a manifestation of the intrinsic nature of GWs :

it is connected to the fact that GWs have only two degrees of freedom represented by the two polarisation states in the TT gauge, and connected to the fact that the graviton has spin 2 and therefore multipoles with $\ell < 2$ do not appear in the multipole decomposition of both the radiation field and the source.

The $\simeq$ sign in $h_{ij}^{TT}(x,t) \simeq \frac{2G}{R} \Lambda_{ij\,kl}\, \ddot{Q}^{kl}(t-r)$ acquires now a new meaning: if we had gone to higher orders in the $\boxed{v \ll 1}$ expansion, we would have found also GW emission from dynamic octupole and all other multiples following it. But the first order in the multipole expansion based on the small velocity expansion that can emit GWs is the QUADRUPOLE. This is also the DOMINANT EMISSION term.

$\boxed{\text{THE QUADRUPOLE FORMULA}}$ concerns the power emitted by a GW source. Let us go back to the expression of the power emitted per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{r^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}^{TT\,ij} \rangle =$$

Eq. (2)

$$[\text{QUAD}] \stackrel{\downarrow}{\simeq} \frac{G}{8\pi} \Lambda_{kl\,mp}(\hat{n}) \langle \dddot{Q}^{kl}(t-r)\, \dddot{Q}^{mp}(t-r) \rangle \text{ and}$$

the total power becomes, with $\int d\Omega\, \Lambda_{klmp} = \frac{2\pi}{15}\left(11\, \delta_{km}\delta_{lp} - 4\, \delta_{kl}\delta_{mp} + \delta_{kp}\delta_{ml}\right)$

$$\boxed{P_{QUAD} = \frac{G}{5} \langle \dddot{Q}_{ij}(t-r)\, \dddot{Q}^{ij}(t-r) \rangle}$$

GW EMISSION FROM A BINARY SYSTEM

the most common example of GW source is given by two compact stellar objects orbiting around each other. We will study the emission from this system, first assuming a Newtonian orbit, and then accounting for the loss of energy and momentum due to the GW emission (leading to inspiral).

$$h_{ij}^{TT}(\underline{x},t) \simeq \frac{2G}{r} \Lambda_{ijkl}(\hat{n}) \ddot{Q}^{kl}(t-r)$$

$$= \frac{2G}{r}\left[(P\ddot{M}P)_{ij} - \frac{1}{2}P_{ij}\, tr(P\ddot{M})\right]$$

REMINDER:

$$\begin{cases} Q^{kl} = M^{kl} - \frac{1}{3}\delta^{kl} M \\ M^{kl} = \int d^3y\, \rho(y) y^k y^l \end{cases}$$

$$\Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$P_{ik} = \delta_{ik} - \hat{n}_i \hat{n}_k$$

Note that M^{kl} and Q^{kl} can be used interchangeably because the trace is always put to zero by Λ_{ijkl}.

let us take <u>the reference frame in which $\underline{x} = \hat{n} r \parallel \hat{z}$</u> (the observer is in the $\hat{z}$ direction)

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad tr(P\ddot{M}) = \ddot{M}_{11} + \ddot{M}_{22}$$

$$P\ddot{M}P = \begin{pmatrix} \ddot{M}_{11} & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & \ddot{M}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the GW can be written as (we are interested in the component propagating in $\hat{n}$ direction)

$$h_{ij}^{TT}(z,t) = \begin{pmatrix} h_+(z,t) & h_\times(z,t) & 0 \\ h_\times(z,t) & -h_+(z,t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

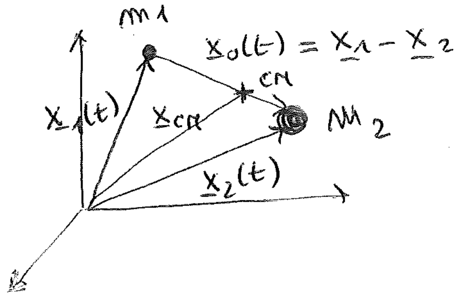
and therefore:

$$(2) \quad \begin{cases} h_+(z,t) = h_{11}(z,t) \simeq \frac{2G}{r}\left[\ddot{M}_{11} - \frac{1}{2}\left(\ddot{M}_{11} + \ddot{M}_{22}\right)\right] = \frac{G}{r}\left(\ddot{M}_{11} - \ddot{M}_{22}\right)(t-r) \\ h_\times(z,t) = h_{12}(z,t) \simeq \frac{2G}{r}\left[\ddot{M}_{12} - \cancel{\frac{1}{2}0}\right] = \frac{2G}{r}\ddot{M}_{12}(t-r) \end{cases}$$

These expressions allow to calculate the amplitudes of a GW in the z-direction generated by a generic distribution of mass. We now specify that the distribution of mass is given by two point particles of masses m_1 and m_2 moving on a circular orbit and isolated, no external forces are acting on them. This is important because it means that $T_{\mu\nu}$ of the source is conserved.

2 MASSES ON CIRCULAR ORBIT DETERMINED SOLELY BY THEIR MUTUAL INTERACTION: their energy momentum tensor is

$$T^{\mu\nu} = m_1 \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta^3(\underline{x}-\underline{x}_1(t)) + m_2 \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta^3(\underline{x}-\underline{x}_2(t))$$



$\underline{x}_{1,2}(t)$ particle's trajectories

$\underline{x}_0(t)$ mutual particle trajectory

$\underline{x}_{CM} = \dfrac{m_1 \underline{x}_1 + m_2 \underline{x}_2}{m_1 + m_2}$ centre of mass position

Let us now choose the centre of mass frame $\underline{x}_{CM} = 0$

$$m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\underline{x}_{CM} = \frac{m_1 \underline{x}_0 + m_1 \underline{x}_2 + m_2 \underline{x}_2}{m} = 0$$

$$\underline{x}_2 = \underline{x}_{CM} - \frac{m_1}{m}\underline{x}_0 = -\frac{m_1}{m}\underline{x}_0$$

$$\underline{x}_1 = \underline{x}_0 + \underline{x}_2 = \underline{x}_0 + \underline{x}_{CM} - \frac{m_1}{m}\underline{x}_0 = \underline{x}_{CM} + \frac{m_2}{m}\underline{x}_0 = \frac{m_2}{m}\underline{x}_0$$

$$M_{ij} = \int d^3x\, T^{00} x_i x_j = m_1 x_{1i} x_{1j} + m_2 x_{2i} x_{2j} =$$

$$= \frac{m_1 m_2^2}{m^2} x_{0i} x_{0j} + \frac{m_2 m_1^2}{m^2} x_{0i} x_{0j}$$

$$= \frac{x_{0i} x_{0j}}{m^2} m_1 m_2 (m_2 + m_1) = \mu\, x_{0i} x_{0j}$$

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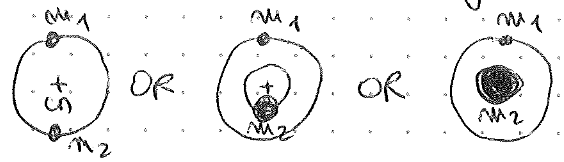
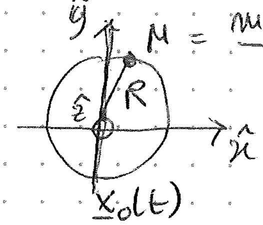
In the centre of mass frame, the second moment (moment of inertia) of a binary system is equivalent to the one of a simple particle with mass corresponding to the reduced mass μ and on a trajectory corresponding to the relative trajectory $\underline{x}_0(t)$ of the two initial masses.

So we can use M_{ij} of a single mass μ on circular orbit:



$\longleftrightarrow$ for the GW emission, these systems are equivalent to



$\mu = \dfrac{m_1 m_2}{m}$

So we have now a circular orbit that is fixed, with radius R and orbital frequency ω_s

$$\begin{cases} x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \\ y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ z_0(t) = 0 \end{cases}$$

The second mass moment is:

$$\begin{cases} M_{11} = \mu\, x_{01}^2 = \mu R^2 \cos^2\left(\omega_s t + \frac{\pi}{2}\right) = \mu R^2 \frac{1}{2}\left(\cos(2\omega_s t + \pi) + 1\right) \\ \quad = \frac{\mu R^2}{2}\left(1 - \cos(2\omega_s t)\right) \\ M_{22} = \mu\, x_{02}^2 = \mu R^2 \sin^2\left(\omega_s t + \frac{\pi}{2}\right) = \mu R^2\left(1 - \cos^2\left(\omega_s t + \frac{\pi}{2}\right)\right) = \\ \quad = \frac{\mu R^2}{2}\left(1 + \cos(2\omega_s t)\right) \\ M_{12} = \mu\, x_{01} x_{02} = \mu R^2 \cos\left(\omega_s t + \frac{\pi}{2}\right)\sin\left(\omega_s t + \frac{\pi}{2}\right) = \\ \quad = \frac{\mu R^2}{2}\sin(2\omega_s t + \pi) = \frac{\mu R^2}{2}\left(-\sin(2\omega_s t)\right) \end{cases}$$

we need to calculate the first three derivatives:

$$\dot M_{11} = \frac{\mu R^2}{2} 2\omega_s \sin(2\omega_s t) \qquad \dot M_{22} = \frac{\mu R^2}{2} 2\omega_s (-\sin(2\omega_s t))$$

$$\ddot M_{11} = \mu R^2 2\omega_s^2 \cos(2\omega_s t) \qquad \ddot M_{22} = \mu R^2 (-2\omega_s^2) \cos(2\omega_s t)$$

$$\dddot M_{11} = -\mu R^2 4\omega_s^3 \sin(2\omega_s t) \qquad \dddot M_{22} = \mu R^2 4\omega_s^3 \sin(2\omega_s t)$$

$$\dot M_{12} = \frac{\mu R^2}{2} 2\omega_s (-\cos(2\omega_s t))$$

$$\ddot M_{12} = \mu R^2 2\omega_s^2 \sin(2\omega_s t)$$

$$\dddot M_{12} = \mu R^2 4\omega_s^3 \cos(2\omega_s t)$$

$\dot M_{11} = -\dot M_{22}$

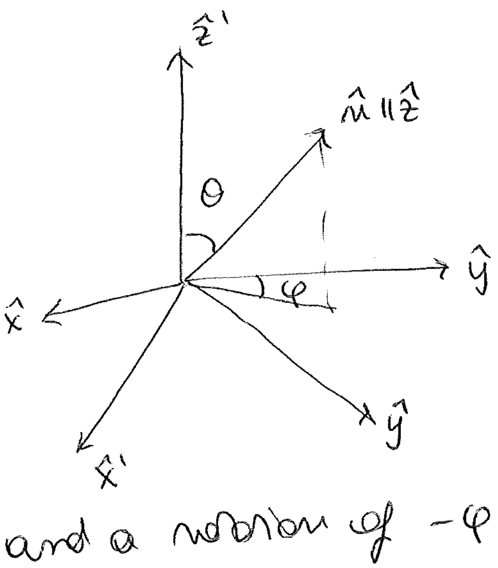
and all other derivatives

Now we can find the two polarisation modes of the GW emitted in the z direction using eq. (2):

$$\begin{cases} h_+(z,t) = \frac{G}{r}(\ddot M_{11} - \ddot M_{22})(t-r) = \frac{G}{r}\mu R^2 4\omega_s^2 \cos(2\omega_s(t-r)) \\ h_\times(z,t) = \frac{G}{r}\ddot M_{12}(t-r) = \frac{G}{r}\mu R^2 2\omega_s^2 \sin(2\omega_s(t-r)) \end{cases}$$

To find the GW emission in a generic direction $\hat n$, we need to rotate the reference frame:

The rotation that brings the frame $(\hat x, \hat y, \hat z)$ in which the observer is in the $\hat z$ direction into the frame $(\hat x', \hat y', \hat z')$ in which the observer is in a generic direction $\hat n$ is



a rotation of $-\theta$ around the $\hat x'$ axis and a rotation of $-\varphi$ around the $\hat z'$ axis:

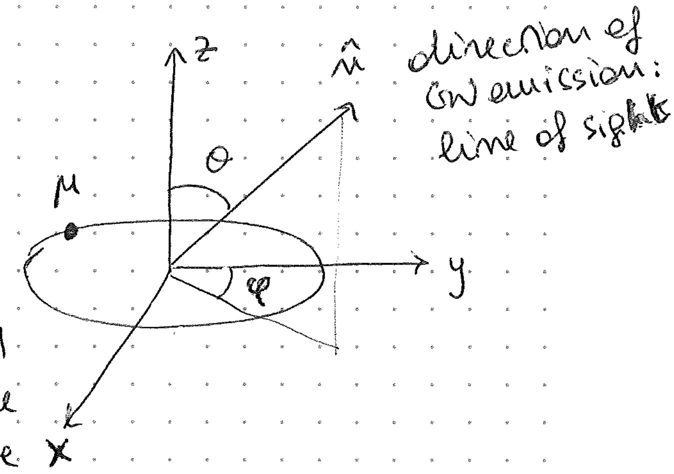
$$R = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

so that the components of the moment of inertia in the $(\hat{x}', \hat{y}', \hat{z}')$ frame are $M'_{ij} = R_{ik} R_{jl} M_{kl} = (R M R^T)_{ij}$

By applying this rotation, one finds the GW polarisation components in the frame in which $\hat{n}$ is a generic direction $\hat{n} = (\theta, \varphi)$ (just given here) (see Maggiore chapter 3.3 and problem 3.2)

$$\begin{cases} h_+(t,\theta,\varphi) = \dfrac{4G}{r} \mu \omega_s^2 R^2 \, \dfrac{1+\cos^2\theta}{2} \cos(2\omega_s t_{ret} + 2\varphi) \\ h_\times(t,\theta,\varphi) = \dfrac{4G}{r} \mu \omega_s^2 R^2 \cos\theta \, \sin(2\omega_s t_{ret} + 2\varphi) \end{cases}$$

$(\hat{x}, \hat{y})$ is the plane of the orbit of the binary, and also the plane of the effective trajectory of the particle μ



∘) From the equations above, we see that a non-relativistic source with characteristic frequency ω_s emits monochromatic quadrupole radiation at frequency $2\omega_s$

∘) The dependence on φ is due to the fact that the system is symmetric under rotation around $\hat{z}$ and a shift in the phase can be reabsorbed by a redefinition of the origin of time

∘) From the degree of polarisation observed, one can deduce the inclination of the orbit of the emitting system:

$\theta = \frac{\pi}{2}$ the orbit is edge-on and there is only + polarization

$\theta = 0$ the system is face-on and the polarization is circular $h_+ = h_\times$

•) the wavelenght of the GWs is such that:

$$\lambda\!\!\!^{-} = \frac{\lambda}{2\pi} = \frac{1}{\omega} = \frac{1}{2\omega_s} \sim \frac{1}{2}\frac{R}{v} >> R$$

$v \sim \omega_s R$

$\omega_s \sim \frac{1}{t_s}$

it is much bigger than the source size when the source is non-relativistic $v << 1$ (the regime of validity of all our calculation). As a consequence, one cannot resolve the GW source by GW observation only, it is a different situation than for EM emission.

•) the TOTAL EMITTED POWER can be calculated using the quadrupole formula: $P_{QUAD} = \frac{G}{5} \langle \dddot{Q}_{ij}(t-r)\dddot{Q}^{ij}(t-r) \rangle$

$$Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij} M = \begin{pmatrix} \frac{2}{3}M_{11} - \frac{1}{3}M_{22} & M_{12} & 0 \\ M_{12} & \frac{2}{3}M_{22} - \frac{1}{3}M_{11} & 0 \\ 0 & 0 & -\frac{1}{3}M \end{pmatrix}$$

back in the coordinate frame in which $\underline{x} = r\hat{n} \parallel \hat{z}$

$(\dddot{M}_{11} = -\dddot{M}_{22})$

$$\dddot{Q}_{ij} = \begin{pmatrix} \dddot{M}_{11} & \dddot{M}_{12} & 0 \\ \dddot{M}_{12} & -\dddot{M}_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\mu R^2 4\omega_s^3 \sin(2\omega_s t) & \mu R^2 4\omega_s^3 \cos(2\omega_s t) & 0 \\ \mu R^2 4\omega_s^3 \cos(2\omega_s t) & \mu R^2 4\omega_s^3 \sin(2\omega_s t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_{QUAD} = \frac{G}{5}\mu^2 R^4 16\,\omega_s^6 \langle\!\langle \sin^2(2\omega_s t) + 2\cos^2(2\omega_s t) + \sin^2(2\omega_s t) \rangle\!\rangle$$

$$= \frac{32}{5} G\mu^2 R^4 \omega_s^6$$

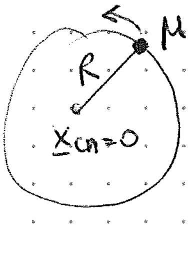
INSPIRAL OF COMPACT BINARIES

In the previous part, we have found the amplidutes of the GWs emitted by a source that is on a stable circular orbit with orbital frequency ω_s and radius R that are fixed. In reality, as the two compact bodies orbit around each other, the GW emission causes the system to loose energy and momentum, inducing a shrinking of the orbit and finally the coalescence. How to account for this, given that in linear theory the energy mom. tensor of the source is conserved? We will do this imposing that the energy lost from the source is equal to the energy contained in the GWs far away from the source. With this trick one can obtain a more realistic description of the GW signal emitted by the binary during the inspiral phase.

However, the final phases of the of the inspiral and the merger cannot be described by linearised theory, not even by doing this trick, as strong field effects come into play and one needs the full GR description.

Also, within linearised theory one cannot account for effects due to the spin or to the finite size of the bodies. However, what we do is enough to have an idea of the signals that have been measured by LVK up to now.

Newtonian dynamics in the cm frame:



$$v = \omega_s R \qquad a = \frac{v^2}{R} = \omega_s v$$

$$a = \frac{v^2}{R} = \frac{F}{\mu} = \frac{G(m_1 m_2)}{R^2 \mu} = \frac{G(m_1+m_2)}{R^2} = \omega_s^2 R$$

$$\boxed{\omega_s^2 = \frac{Gm}{R^3}}$$

the total energy of the binary is:

$$E_{kin} + E_{pot} = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{R} \underset{\uparrow}{=} -\frac{Gm_1m_2}{2R} = E_{orbit}$$

$$v^2 = \frac{Gm}{R}$$

negative energy because the orbit is bound

1) the total energy of the system must diminish due to the GW emission, so R must diminish (- sign in the energy)

2) if R diminishes, ω_s grows

3) if ω_s grows, the emitted power $P_{QUAD} = \frac{32}{5} G\mu^2 R^4 \omega_s^6$ grows as well

4) if the emitted power grows, R shrinks even more

$\Rightarrow$ THE RESULT OF THIS RUNAWAY PROCESS IS THE COALESCENCE OF THE BINARY SYSTEM

let us analyse the system in the approximation that the orbit remains circular with a slowly varying radius: $\boxed{|\dot{R}| \ll v}$

this gives then: $R \propto \omega_s^{-2/3}$ $\quad \dot{R} \propto \omega_s^{-5/3}\dot{\omega}_s \underset{\uparrow}{\ll} \omega_s^{1/3} \Rightarrow \boxed{\dot{\omega}_s \ll \omega_s^2}$

$v = \omega_s R$

let us now impose that the ENERGY LOST FROM THE SYSTEM is EQUAL TO THE ENERGY EMITTED IN GWs FAR AWAY FROM THE SOURCE:

$$\boxed{-\frac{dE_{orbit}}{dt} = P_{QUAD}}$$

$$-\frac{dE_{orbit}}{dt} = \frac{d}{dt}\left(\mu m \frac{G}{2R}\right) = \frac{d}{dt}\left(\mu m \frac{G}{2} \frac{\omega_s^{2/3}}{(Gm)^{1/3}}\right) = \frac{\mu}{2}(Gm)^{2/3}\frac{2}{3}\frac{\dot{\omega}_s}{\omega_s^{1/3}}$$

$$= P_{QUAD} = \frac{32}{5} G\mu^2 \frac{(Gm)^{4/3}}{\omega_s^{8/3}} \omega_s^6$$

$$\Rightarrow \boxed{\dot{\omega}_s = \frac{96}{5}\mu\, G^{5/3}\, m^{2/3}\, \omega_s^{11/3}}$$

this is the time variation of the orbital frequency due to the GW emission that causes the orbit to loose energy. If we now rewrite this in terms of the GW frequency

$f_{GW} = \frac{\omega_{GW}}{2\pi} = \frac{\omega_s}{\pi}$ (GW frequency is twice the orbital one)

$$\boxed{\dot{f}_{GW} = \frac{96}{5}\pi^{8/3}(G M_c)^{5/3} f_{GW}^{11/3}}$$

with the CHIRP MASS defined as $\boxed{M_c = \mu^{3/5} m^{2/5}}$

This equation is of fundamental importance, since GW detectors can measure both the frequency of the GW signal and how it varies with time, and therefore with these measurements they can obtain the chirp mass of the binary system.

In order to find the separate masses, one needs to resort to full GR effects: in linearised theory, all quantities depend solely on the combination giving the chirp mass.

This equation describes how the GW frequency increases in time due to the shrinking of the orbit radius because of the GW emission itself.

However, it has been obtained in linearised theory under the assumption $\dot{\omega}_s \ll \omega_s^2$. It breaks down in the last stages of the inspiral and at merger, which require full GR to be described (actually, numerical simulations of the system in GR).

we can solve the equation setting $\tau = t_{coal} - t$, where t_{coal} denotes the time after which the equation is no longer valid and the binary is about to coalesce. We assign infinite frequency to that time, so that:

$$\int_{f_{GW}}^{\infty} df' \frac{5}{96\pi^{8/3}} \frac{1}{(GM_c)^{5/3}} f^{-\frac{11}{3}} = \int_{t}^{t_{coal}} dt'$$

$$\frac{5}{96\pi^{8/3}} \frac{1}{GM_c^{5/3}} \left[0 - \frac{f_{GW}^{-8/3}}{-8/3} \right] = \tau$$

$$\Rightarrow \boxed{f_{GW}(\tau) = \left(\frac{5}{256}\right)^{3/8} \frac{(GM_c)^{-5/8}}{\pi} \left(\frac{1}{\tau}\right)^{3/8}}$$

the GW frequency depends on the stage at which the binary is, meaning how far it is from coalescence.

therefore, different detectors, operating at different frequencies, can detect the GW emission from different kinds of binaries at different stages of the binary evolution.

let us rewrite this equation in a more informative way restoring dimensions

$$f_{GW} = \left(\frac{5}{256}\right)^{3/8} \frac{1}{\pi} \left(\frac{c^3}{GM_c}\right)^{5/8} \left(\frac{1}{\tau}\right)^{3/8}$$

$$\frac{f_{GW}}{100\,Hz} = \frac{sec}{100} \left(\frac{5}{256}\right)^{3/8} \frac{1}{\pi} \left(\frac{M_\odot}{M_c}\right)^{5/8} \left(\frac{27 \cdot 10^{15}}{1.3 \cdot 10^{11}}\right)^{5/8} \frac{km^{15/8}}{sec^{15/8}} \frac{sec^{10/8}}{km^{15/8}} \frac{1}{sec^{3/8}} \left(\frac{sec}{\tau}\right)^{3/8}$$

$$G = 1.3 \cdot 10^{11} \frac{km^3}{M_\odot\, sec^2}$$

$$c = 3 \cdot 10^5 \frac{km}{sec}$$

$$\boxed{\frac{f_{GW}}{100\,Hz} \simeq 1.5 \left(\frac{M_\odot}{M_c}\right)^{5/8} \left(\frac{sec}{\tau}\right)^{3/8}}$$

From this equation, one already sees that LVK devices can observe binaries of the order of a few solar masses and they are very close to coalescence:

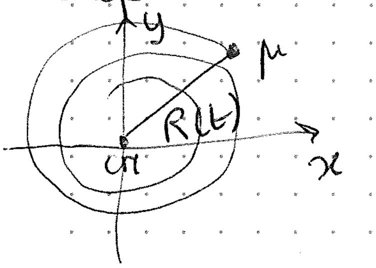
- first LVK detection of black hole binary: $M_c \simeq 28\, M_\odot$, $\tau \simeq 0.2$ sec, $f_{GW} \simeq 35$ Hz
- first LVK detection of NS binary: $M_c \simeq 1.2\, M_\odot$, $\tau \simeq 30$ sec, $f_{GW} \simeq 38$ Hz
- the space interferometer LISA can detect GWs at frequency of about $10^{-3} - 10^{-2}$ Hz:

$$M_c = 25\, M_\odot \quad \tau = 10 \text{ years} \quad f_{GW} = 10^{-2}\text{ Hz}$$
$$M_c = 10^6\, M_\odot \quad \tau = 1 \text{ hour} \quad f_{GW} = 10^{-3}\text{ Hz}$$

- Pulsar timing arrays detect at much lower frequencies, around 10^{-9} Hz

$$M_c = 10^9\, M_\odot \quad \tau = 10^5 \text{ years} \quad f_{GW} = 7 \cdot 10^{-9}\text{ Hz}$$

We have seen the frequency, what about the amplitude? With respect to what we did in the previous part, the GW amplitude as well is affected by the fact that one must account for the back-reaction of the GW emission on the source. The orbital radius and angular velocity now depend on time, so the new orbit is:



$$\begin{cases} x_0(t) = R(t) \cos\left(\frac{\phi(t)}{2}\right) \\ y_0(t) = R(t) \sin\left(\frac{\phi(t)}{2}\right) \end{cases}$$

where

$$\phi(t) = 2 \int_{t_0}^{t} dt' \, \omega_s(t')$$

new phase that depends on time

In principle one needs to redo the calculation of $\ddot{M}_{ij}$ with the new time-dependent orbit, and one gets terms depending on both $\dot{R}$ and $\dot{\omega}_s$. These however can be dropped under the assumption that we are doing:
$$\begin{cases} \dot{\omega}_s \ll \omega_s^2 \\ \ddot{R} \ll R\omega_s = v \end{cases}$$

we can therefore simply substitute in the eqs for h_+ and $h_\times$ derived before:
$$G\mu\omega_s^2 R^2 = G\mu\omega_s^2 \frac{(Gm)^{2/3}}{\omega_s^{4/3}} = (GM_c)^{5/3}\omega_s^{2/3}$$

$f_{GW} = \omega_s/\pi$

$$\begin{cases} h_+(t) = \frac{4}{R}(GM_c)^{5/3}(\pi f_{GW}(t_{ret}))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi(t_{ret})) \\ h_\times(t) = \frac{4}{R}(GM_c)^{5/3}(\pi f_{GW}(t_{ret}))^{2/3}\cos\theta\sin(\phi(t_{ret})) \end{cases}$$

where the phase is given by:
$$\phi(\tau) = 2\int_\tau^0 d\tau'\left(\frac{5}{256}\right)^{3/8}\frac{1}{(GM_c)^{5/8}}\left(\frac{1}{\tau'}\right)^{3/8} = \frac{2}{(5GM_c)^{5/8}}\left[0-\tau^{5/8}\right]$$

$\omega_s = \pi f_{GW}$

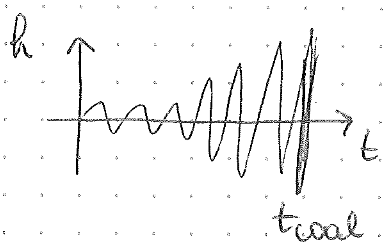
$$\boxed{\phi(\tau) = -\frac{2}{(5GM_c)^{5/8}}\tau^{5/8}}$$

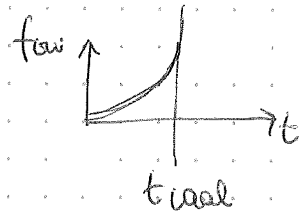
Therefore, putting it all together, we have the following behaviour as the binary approaches coalescence and its orbit shrinks:

$\tau = t_{coal} - t$ decreases $\rightarrow$ $f_{GW} \propto \tau^{-3/8}$ increases

$\rightarrow$ $h_{+,\times} \propto f_{GW}^{2/3}$ increase and $\phi(\tau) \propto -\tau^{5/8}$ decrease

This particular behaviour is called the <u>CHIRP SIGNAL</u>





Linearised theory manages to describe the gross features of the signal.

Again, for the first LVK detection $\begin{cases} M_c \simeq 28\, M_\odot \\ r \simeq 410\ \mathrm{Mpc} \\ f_{GW} \simeq 35\ \mathrm{Hz} \end{cases}$

$$h \simeq \frac{4}{410\ \mathrm{Mpc}} \frac{(G M_\odot)^{5/3}}{c^{12/3}} \left(\frac{28 \cancel{M_\odot}}{\cancel{M_\odot}}\right)^{5/3} (\pi f_{GW})^{2/3} \simeq$$

$$\simeq \frac{4}{410} \frac{1}{3\cdot 10^{19}\ \cancel{\mathrm{km}}} \frac{(1.3\cdot 10^{11})^{5/3}}{(3\cdot 10^{5})^{4}} \frac{\cancel{\mathrm{km}^5}\ \cancel{\mathrm{sec}^{-10/3}}}{\cancel{\mathrm{km}^4}\ \cancel{\mathrm{sec}^{-4}}} (28)^{5/3} \pi^{2/3} (35)^{2/3}\ \mathrm{sec}^{-2/3}$$

$\simeq 10^{-21}$ we find the same amplitude as anticipated in the chapter about GW detectors

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GRAVITATIONAL LENSING

We have seen in the section of nearly Newtonian fields that light travels on null geodesics, and that the latter aren't straight lines if the metric is non trivial. Indeed, we had derived a dispersion relation of the kind:

$$(\nabla u(x))^2 \simeq (1-4\phi)\,\omega^2 \quad \text{for a light ray } A_\mu = C_\mu e^{iS} \text{ with } S = u(\underline{x}) - \omega t$$

that shows that light rays in this nearly-Newtonian setting travel as if in a inhomogeneous medium with refractive index $n = \sqrt{1-4\phi}$.

The fact that light rays travel on curved geodesics causes a change in the apparent position of the source, and can also magnify and distort the source shape. This can be used to test the metric between the source and the observer (for example, to infer the presence of dark matter or dark energy).

LENSING AND TIME DELAY IN NEARLY-NEWTONIAN SETTING

The metric is:

$$\begin{cases} g_{00} = -(1+2\phi) \\ g_{0i} = 0 \\ g_{ij} = (1-2\phi)\,\delta_{ij} \end{cases}$$

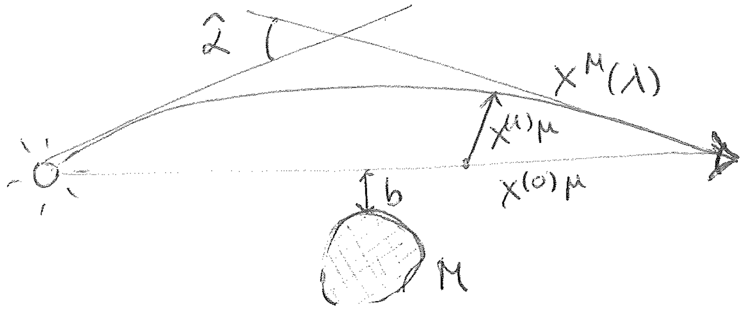
and ϕ is static and satisfies: $\boxed{\Delta\phi = 4\pi G\rho}$

The Christoffel symbols in linearised theory are

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}\right)$$

$$\Gamma^0_{0i} = \partial_i\phi = \Gamma^0_{i0} = \Gamma^i_{00} \qquad \Gamma^i_{jk} = -\delta_{ij}\partial_k\phi - \delta_{ik}\partial_j\phi + \delta_{jk}\partial_i\phi$$

We now analyse the following set-up:



$$\underbrace{x^\mu(\lambda)}_{\text{photon geodesic}} = \underbrace{x^{(0)\mu}(\lambda)}_{\text{background path}} + \underbrace{x^{(1)\mu}(\lambda)}_{\text{perturbation}}$$

$x^{(0)\mu}(\lambda)$ describes the path in the background space-time, that is $\eta_{\mu\nu}$: <u>straight null path</u>

$x^{(1)\mu}(\lambda)$ deviation between the true and bckg geodesic

This remains a small perturbation, and we can evaluate all quantities on the bck path $x^{(0)\mu}$, as long as the metric along the bckg and the perturbed paths isn't too different: $\underbrace{x^{(1)i}\,\partial_i\phi}_{\text{how the metric changes along } x^{(1)\mu}} \ll \phi$

background null wave vector $k_\mu k^\mu = 0$: $k^\mu = \dfrac{dx^{(0)\mu}}{d\lambda} \rightarrow (k^0)^2 = |\underline{k}|^2$

derivative of the deviation vector: $\ell^\mu = \dfrac{dx^{(1)\mu}}{d\lambda}$

the full trajectory is null: $g_{\mu\nu} \dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} = 0$

the full trajectory is a geodesic : $\dfrac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{d\lambda}\dfrac{dx^\beta}{d\lambda} = 0$

the null trajectory condition gives at first order in perturbation:

$$(\eta_{\mu\nu} + h_{\mu\nu})(k^\mu k^\nu + k^\mu \ell^\nu + k^\nu \ell^\mu) =$$

$$= 0 + 2\eta_{\mu\nu} k^\mu \ell^\nu + h_{\mu\nu} k^\mu k^\nu = 0$$

$$2 k \ell^0 + 2\, \underline{k} \cdot \underline{\ell} + 2\phi k^2 + 2\phi |\underline{k}|^2 = 0$$

$$\Rightarrow \quad k\ell^0 + \underline{k} \cdot \underline{\ell} = 2\phi k^2$$

the geodesic condition for the full trajectory:

- bckg: $\frac{d^2 x^{(0)\mu}}{d\lambda^2} = 0$ → the bckg. trajectory is a straight path

- I order in perturbation: $\frac{d^2 x^{(1)\mu}}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^{(0)\alpha}}{d\lambda} \frac{dx^{(0)\beta}}{d\lambda} = 0$

this is pure perturbation

$$\frac{d\ell^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0$$

$$\mu = 0: \quad \frac{d\ell^0}{d\lambda} + 2k\, \partial_i \phi\, k^i = \frac{d\ell^0}{d\lambda} + 2k(\underline{k} \cdot \nabla\phi) = 0$$

$$\mu = i: \quad \frac{d\underline{\ell}}{d\lambda} + k^2 \nabla\phi - 2k_i(\underline{k} \cdot \nabla\phi) + k^2 \nabla\phi =$$

$$= \frac{d\underline{\ell}}{d\lambda} + 2k^2\left[\nabla\phi - \hat{k}(\hat{k} \cdot \nabla\phi)\right] = 0$$

$$\frac{d\underline{\ell}}{d\lambda} = -2k^2 \nabla_\perp \phi$$

where the gradient of the potential in the direction perpendicular to the propagation direction is

$$\nabla_\perp \phi = \nabla\phi - \nabla_{\parallel}\phi = \nabla\phi - \hat{k}(\hat{k}\cdot\nabla\phi)$$

We integrate the $\mu = 0$ geodesic:

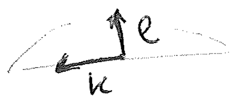
$$e^0 = \int \frac{de^0}{d\lambda} d\lambda = \int -2k\,(\underline{k}\cdot\nabla\phi)\,d\lambda = -2k\int \frac{d\underline{x}^{(0)}}{d\lambda}\cdot\nabla\phi\, d\lambda =$$

$$= -2k\int \nabla\phi\cdot d\underline{x}^{(0)} = -2k\phi$$

setting $e^{(0)} = 0$ for $\phi = 0$

and therefore from $ke^0 + \underline{k}\cdot\underline{e} = 2\phi k^2 \Rightarrow \underline{k}\cdot\underline{e} = 0$

the derivative of the deviation vector is orthogonal to the unperturbed wave-vector at first order.



Furthermore, the rate of change of the deviation vector is due to the change in the potential perpendicular to the back trajectory

$$\frac{d\underline{e}}{d\lambda} = -2k^2\,\nabla_\perp\phi$$

total change in the deviation vector $\Leftarrow$ $\Delta\underline{e} = \int \frac{d\underline{e}}{d\lambda} d\lambda = -2k^2\int \nabla_\perp\phi\, d\lambda$

One can define the DEFLECTION ANGLE $\hat{\alpha}$ as the amount by which the original spatial wave-vector is deflected as it travels from a source to the observer: this angle is a two-dimensional vector in the plane orthogonal to the wave vector $\underline{k}$:

$$\hat{\alpha} = -\frac{\Delta\underline{e}}{k}$$

the observer is looking backwards with respect to the photon propagation and therefore, defining $s = k\lambda$ as physical spatial distance that has been travelled $\frac{ds}{d\lambda} = \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} = k$

one can rewrite the deflection angle as:

$$\boxed{\hat{\alpha} = 2\int \nabla_\perp \phi \, ds}$$

In addition to a deflection of the light path there is also a gravitational time delay:

SHAPIRO TIME DELAY: the presence of a Newtonian potential along the photon path slows down the photons with respect to the case without potential

let us consider that the observer is far away from the mass generating the potential, and that it is at rest in the inertial bckg: then we can approximate coordinate time t with proper time at the observer: $\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \simeq t$

total time elapsed along null path:

$$t = \int \frac{dx^0}{d\lambda} d\lambda$$

Time delay due to the perturbed trajectory:

$$\Delta t = \int \frac{dx^{(1)0}}{d\lambda} d\lambda = \int \ell^0 d\lambda = -2k \int \phi \, d\lambda$$

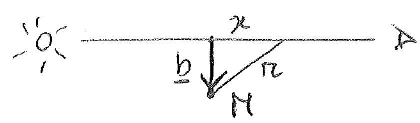
$$= -2\int \phi \, ds \qquad \text{(s physical spatial distance)}$$

All these predictions have been tested by experiments and confirmed. Notably light deflection by the sun during a total solar eclipse by Eddington in 1919.

We now calculate what we have been deriving in the specific case of a $\boxed{\text{point mass}}$

Point mass with impact parameter b: ϕ along the background trajectory is:

$$\phi = -\frac{GM}{r} = -\frac{GM}{\sqrt{x^2+b^2}}$$



transverse gradient $\nabla_\perp \phi = \frac{GM}{(x^2+b^2)^{3/2}} \underline{b}$

along the direction perpendicular to photon path

deflection: $$\hat{\alpha} = 2GMb \int_{-\infty}^{+\infty} \frac{dx}{(x^2+b^2)^{3/2}} = \frac{4GM}{b}$$

assume light source and observer far away

For the sun: $$\hat{\alpha} = \frac{4GM_\odot}{c^2 R_\odot} = \frac{4\,(1.3\cdot10^{11})}{(3\cdot10^5)^2\ 6.96\cdot10^5} \frac{km^3}{sec^2}\frac{sec^2}{km^2}\frac{1}{km} =$$

$$GM_\odot = 1.3\cdot10^{11}\ \frac{km^3}{sec^2}$$

$$R_\odot = 6.96\cdot10\ km$$

$$c = 3\cdot10^5\ \frac{km}{sec}$$

$$= 8.3\cdot10^{-6}\ rad \simeq 1.7\ arc\ sec$$