Weak Gravitational Fields

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1 Heuristic approach to the Newtonian metric

You have demonstrated in the lectures that, for weak gravitational fields, the metric reads at lowest order

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + (1-2\Phi)\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{1}$$

where Φ is the Newtonian gravitational potential. What does not trivially come out of the rigorous derivation of the above line-element is the physical meaning of the global coordinates (t, x^i). In particular, they are *not* some Riemann of Fermi normal coordinates, and they do not coincide with the times and distances measured by a static observer ($x^i = \mathrm{cst}$) in that space-time.

This short exercise aims to determine the nature of (t, x^i) from a *Gedankenexperiment* and simple arguments following from Einstein's equivalence principle. For simplicity, we will restrict to a one-dimensional situation, and focus on the coordinates (t, x)

Consider the following situation (see fig. 1): an experimentalist, Alice, goes on the top of a high building made of many identical floors of thickness h. On her way up, she leaves on each floor a clock ticking every second; she keeps a clock for herself and a ruler. We take the top of the building as the zero-point of the Newtonian potential: $\Phi_{\rm top} = 0$. Let E be an arbitrary event happening in the building—e.g., a door slams. We define its coordinates t, x as follows: Alice jumps from the top of the building so that she reaches the door's floor at the moment when it slams; t is defined as the time indicated on her own clock, and x is the distance that she has fallen *as measured in her own frame.*

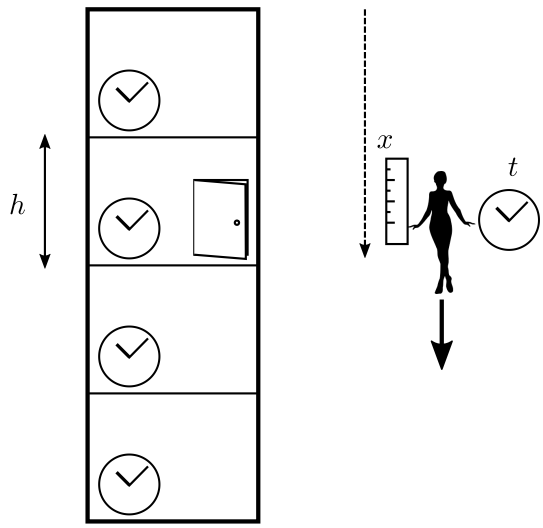

Figure 1 The coordinates x, t are defined by the freely-falling observer.

Q1. Let two events E_1, E_2 be the successive ticks ($\delta\tau = 1\,\mathrm{s}$) of a clock on a floor where the potential is Φ. Using only special relativity and Newtonian dynamics, show that

$$\delta t \equiv t_2 - t_1 = \frac{\delta\tau}{\sqrt{1+2\Phi}}, \tag{2}$$

and conclude that $g_{tt} = -(1+2\Phi)$.

Q2. Show that the thickness of this floor as measured by Alice is

$$\delta x = h\sqrt{1+2\Phi} \ , \tag{3}$$

and deduce from it that $g_{xx} = (1+2\Phi)^{-1} \approx (1-2\Phi)$. On Earth, what is the order of magnitude of the mistake that we make with this latter approximation? [Hint: Use that Alice measures the distance between two doors which appear to slam at the exact same moment in her frame.]

Q3. Considering the emission and reception of light from the point of view of a static observer in the building, show simply that $g_{tx} = 0$.

2 Gravito-magnetism

This exercise is dedicated to a post-Newtonian feature of relativistic gravitation, namely *gravito-magnetism*. Recall that, if the metric is almost Minkowskian, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, then the linearised Einstein's equation reads,

$$\Box\gamma_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \qquad \gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\, h\, \eta_{\mu\nu} \ , \tag{4}$$

in Hilbert's gauge $\partial_\mu \gamma^{\mu\nu} = 0$. We assume that matter is made of non-relativistic massive objects, which implies that $T_{00} \gg T_{0i} \gg T_{ij}$. In this exercise, we will neglect T_{ij} but keep T_{0i}.

Q1. Introducing $A_\mu \equiv \gamma_{0\mu}/4$ and $J_\mu \equiv G T_{0\mu}$, show that the linearised Einstein equation is analogous to relativistic electrodynamics. What plays the role of the electric charge? electric currents? Discuss the presence of a perhaps unexpected minus sign.

Q2. Show that, at this post-Newtonian order, the metric reads

$$\mathrm{d}s^2 = -(1 - 2A_0)\mathrm{d}t^2 + 8A_i \mathrm{d}x^i \mathrm{d}t + (1 + 2A_0)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \ . \tag{5}$$

In the post-Newtonian expansion scheme, the small parameter is the typical velocity of sources, $\varepsilon \sim v$. We also assume that the source of the gravitational fields is 'almost' stationary, i.e. the time derivatives are much smaller than spatial derivatives so that $\partial_t \sim \varepsilon \partial_i$.

Q3. Expanding the geodesic equation at linear order in the metric and in ε, show that freely-falling particles satisfy the following equation of motion:

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = E^i_{\mathrm{g}} + \varepsilon^i{}_{jk} v^j B^k_{\mathrm{g}} \ , \tag{6}$$

with $v^i = \mathrm{d}x^i/\mathrm{d}t$, $v^2 \equiv \delta_{ij} v^i v^j$, and where you will give the expression of $E^i_{\mathrm{g}}, B^i_{\mathrm{g}}$ in terms of A_μ. What is the difference with electrodynamics?

[N.B. This exercise has been slightly modified/simplified, the original formulation required to go up to 2nd order in ε.]

Consider a galaxy of the Milky Way type, which we model as a thick disk of radius $R = 20\,\mathrm{kpc}$, and thickness $h = R/100 = 0.2\,\mathrm{kpc}$. For simplicity, we will assume this disk to be homogeneous, with density $0.1 M_\odot/\mathrm{pc}^3$, and rigidly rotating with angular velocity $\Omega = 10^{-15}$ rad/s.

Q4. From your knowledge of electro-statics and magneto-statics, give the integral expressions of the gravito-electric and gravito-magnetic fields created by the galaxy. Integrate numerically those results, and compare the gravito-magnetic force with the gravito-electric (Newtonian) force experienced by a star located at the outskirts of the galaxy.