

Fermi Normal Coordinates

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The first version of this exercise sheet has been proposed by Dr Pierre Fleury in the 2018/2019 tutorial for the GR class. We warmly thank Pierre for his work!

This exercise is dedicated to a particularly interesting coordinate system, namely *Fermi normal coordinates*, which form the mathematical basis of the equivalence principle. Its physical meaning is the following. Suppose that you are freely falling and non-rotating, and that you carry a clock, along with a set of three long rigid rulers which you hold perpendicular to each other. The four coordinates $(X^\alpha) \equiv (\tau, X, Y, Z)$, where τ is the time measured with the clock, and X, Y, Z the lengths measured with the rulers, form your Fermi normal coordinates.

Reference. This exercise is adapted from the book *A Relativist's Toolkit*, by Eric Poisson.

1 The observer's tetrad

Consider an arbitrary spacetime (\mathcal{M}, g) provided with an arbitrary coordinate system $\{x^\mu\}$. Let \mathcal{L} be the worldline of a freely falling observer.

Q1. What can you say about \mathcal{L} ?

At a given point $A \in \mathcal{L}$, we consider a *tetrad*, i.e. a set of four four-vectors (e_α) such that

$$g(e_\alpha, e_\beta)|_A = \eta_{\alpha\beta}. \quad (1)$$

Q2. How does one usually call such a set of vectors in geometry?

Let us further assume that $e_0 = \mathbf{u}$, the observer's four-velocity at A , and let us extend e_α on the whole \mathcal{L} by parallel transporting them from A to any other point $B \in \mathcal{L}$.

Q3. Justify that eq. (1) holds at every point of \mathcal{L} .

The tetrad field $\{e_\alpha\}$, now defined all along \mathcal{L} , will constitute the axes of the Fermi normal coordinate system centred on the observer.

2 Construction of the coordinates

Let Q be any point in \mathcal{M} . Assume that there is a unique space-like geodesic \mathcal{G} which connects this point to \mathcal{L} orthogonally, in the sense that the (unit) tangent vector \mathbf{v} of \mathcal{G} is orthogonal to $\mathbf{u} = e_0$ at $P = \mathcal{L} \cap \mathcal{G}$, that is $g(\mathbf{v}, e_0)|_P = 0$.

Q4. Show that, at P , there exist three numbers $(v^a)_{a=1,2,3}$ such that $\mathbf{v} = v^a e_a$.

We now define the geodesic distance s from P to Q as

$$s = \int_P^Q ds = \int_P^Q \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \quad (2)$$

along the geodesic \mathcal{G} , where λ can be any parameter along it. The Fermi normal coordinates of Q are then defined as (τ, X^a) , with τ the proper time of P , with respect to an arbitrary origin $O \in \mathcal{L}$ where $\tau = 0$; and with

$$X^a \equiv s v^a. \quad (3)$$

Q5. Summarise this construction with a drawing depicting $\mathcal{L}, Q, \mathcal{G}, P, (\mathbf{e}_\alpha), \mathbf{v}, s, O, \tau$.

3 Metric around \mathcal{L}

Suppose that we want to perform the transformation from the arbitrary coordinate system $\{x^\mu\}$ to the Fermi coordinates $\{X^\alpha\}$. This consists of four functions $x^\mu(\tau, X^a = s v^a)$.

Q6. Justify that $\partial x^\mu / \partial \tau|_{X^a=0} = e_0^\mu$ and $\partial x^\mu / \partial s|_{s=0} = v^\mu|_{s=0}$.

Q7. Deduce that, on \mathcal{L} , we have $\partial x^\mu / \partial X^\alpha = e_\alpha^\mu$. What do you conclude about $g_{\alpha\beta}(\tau, X^a = 0)$?

We now wish to determine the first derivatives of the metric on \mathcal{L} .

Q8. Using that the curves defined by $s \mapsto x^\mu(\tau, v^a s)$, for fixed values of τ and v^a , are geodesics, show that $\Gamma_{bc}^\alpha = 0$.

Q9. Demonstrate that the remaining coefficients, namely $\Gamma_{\tau\tau}^\alpha$ and $\Gamma_{\tau b}^\alpha$ vanish because of the parallel transport condition on the observer's tetrad. Deduce that

$$\partial_\alpha g_{\beta\gamma}(\tau, X^a = 0) = 0. \quad (4)$$

Q10. Show that, on \mathcal{L} , $g_{\alpha\beta,\gamma\tau} = 0$ and $g_{\tau\tau,ab} = -2R_{\tau a \tau b}$.

It is possible to show that the other second derivatives of the metric on \mathcal{L} can be expressed in terms of the Riemann tensor. However, the calculations are slightly more involved, so that you can *accept* the following relations without proving them:

$$g_{\tau a, bc} = -\frac{2}{3}(R_{\tau bac} + R_{\tau cab}), \quad (5)$$

$$g_{ab, cd} = -\frac{1}{3}(R_{acbd} + R_{adbc}). \quad (6)$$

Q11. Conclude by giving the expression of the metric, in terms of Fermi coordinates around \mathcal{L} , up to second order in X^a .