Nordström's Theory of Gravity

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Gunnar Nordström

Proposed in 1912, that is, only three years before Einstein's general relativity (GR), Nordström's theory of gravity is probably the first consistent relativistic theory of gravitation ever proposed. Its geometric reformulation by Einstein and Fokker in 1914 definitely paved the way towards GR, which shows that the birth of the latter has been the fruit of a collective research effort. Although it turned out to be an incorrect description of nature, Nordström's theory still has an interesting pedagogical interest, in particular for the understanding of some extensions of GR, known as scalar-tensor theories.

1 Original formulation

In its original form, Nordström's theory is technically very close to any relativistic field theory, like electrodynamics. The idea consists in considering Newton's gravitational field Φ as a scalar field in Minkowski spacetime, conformally coupled to matter. If we assume that matter is made of N massive point-particles, then the action of the system reads

$$S = -\frac{1}{8\pi G}\int \mathrm{d}^4x \; \eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \sum_{a=1}^{N} S_a[\boldsymbol{y}_a, \Phi], \tag{1}$$

where $\{x^\mu\}$ is an inertial coordinate system, $\eta_{\mu\nu}$ is the Minkowski metric, and S_a is the action of a point particle, with a slight modification:

$$S_a[\boldsymbol{y}_a, \Phi] = -m_a \int \mathrm{d}\tau_a \; [1 + \Phi(\boldsymbol{y}_a)]. \tag{2}$$

In the above, $\boldsymbol{y}_a$ denote the trajectory of particle a, τ_a its proper time, m_a its mass, and $(1+\Phi)$, called conformal factor, will be responsible for the coupling between matter and gravitation[1].

Q1. Using a similar trick as last week, show that the action of a single particle can be rewritten

$$S_a = -\int \mathrm{d}^4x \; (1+\Phi)\,\gamma m_a(1-v^2)\delta_{\mathrm{D}}[x^i - y_a^i(t)], \tag{3}$$

where v is the velocity of the particle, and γ the corresponding Lorentz factor. Conclude that the total matter action reads

$$\sum_{a=1}^{N} S_a[\boldsymbol{y}_a, \Phi] = -\int \mathrm{d}^4x \; (1+\Phi)(\rho - 3P), \tag{4}$$

where you will give the expression and physical interpretation of ρ and P.

[1] This should remind you of electrodynamics, where charged particles have a coupling of this form, Φ being replaced by $qu^\mu A_\mu$, q being the charge of the particle and $\boldsymbol{u}$ its four-velocity.

Q2. Taking the derivative of the total action S with respect to Φ, show that the associated field equation reads

$$\Box\Phi = 4\pi G(\rho - 3P) \ , \tag{5}$$

with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. In which limit is it equivalent to the Poisson equation of Newtonian gravity? Which new phenomena can it describe?

The above shows how matter generates a gravitational field. Let us now see how this field can, in turn, affect the motion of matter. For that purpose, let us consider only one of the particle described by the action (2). We will drop the subscripts a to alleviate notation. If the trajectory of the particle is parametrised by an arbitrary λ, this action thus reads

$$S_1[\boldsymbol{x}] = -m \int \mathrm{d}\lambda \, \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \, [1 + \Phi(\boldsymbol{x}(\lambda))]. \tag{6}$$

Q3. Show that equation of motion of the particle dictated by the action (6) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(1+\Phi)u^\mu\right] = -\partial^\mu\Phi \tag{7}$$

Q4. In which regime do we recover Newton's second law?

Q5. Consider the following situation: there exists a frame in which Φ is homogeneous in space, but evolving with time $\Phi(t)$. What is the corresponding gravitational force? Does this happen in Newtonian gravity?

2 The Einstein-Fokker formulation

In 1914, Einstein and Fokker found that Nordström's gravity could be reformulated in terms of a curved space-time. This was a dramatic change of paradigm: instead of viewing gravity as a force, mediated by a field in Minkowski space-time, as in Nordström's original formulation, it started to be seen as a distortion of the geometry of space and time.

Consider the metric defined by

$$g_{\mu\nu} = (1+\Phi)^2\eta_{\mu\nu}, \tag{8}$$

where $\{x^\mu\}$ is still assumed to be an inertial coordinate system with respect to $\eta_{\mu\nu}$, for simplicity.[2]

Q6. Show that the Christoffel symbols of $g_{\mu\nu}$ are

$$\Gamma^\rho{}_{\mu\nu} = \left(\delta^\rho_\mu\partial_\nu + \delta^\rho_\nu\partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\partial_\sigma\right)\ln(1+\Phi). \tag{9}$$

Q7. From the above, show that the Ricci scalar reads $R = -6(1+\Phi)^{-3}\Box\Phi$, with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Q8. Demonstrate that the action (6) for a point particle is equivalent to its general-relativistic counterpart, with the metric $\boldsymbol{g}$. What can you say about free fall in Nordström's theory?

Q9. Using the results of a previous exercise sheet, give the expression of the corresponding stress-energy tensor, and conclude that the quantity $\rho - 3P$ introduced in eq. (5) reads

$$\rho - 3P = -(1+\Phi)^3 T, \tag{10}$$

where $T = g_{\mu\nu}T^{\mu\nu}$ is the trace of the total stress-energy tensor of the system of N particles.

Q10. How can you rewrite Nordström's equation (5) in purely geometric terms? Compare with Einstein's equation.

[2] A more general formulation would be to write $\boldsymbol{g} = (1+\Phi)^2\boldsymbol{f}$, where $\boldsymbol{f}$ would denote the flat Minkowski metric, but we do not need such a generality here.