Stress-Energy Tensor

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You have seen in the course that the energy and momentum of a matter field is described by its stress-energy tensor $\boldsymbol{T}$. This tensor typically encodes information about the energy density, the energy flux, and the momentum flux of matter. An important property of $\boldsymbol{T}$ is that it can be extracted directly from the action of matter. More precisely, consider a matter species—which we will represent here by a field $\boldsymbol{\psi}$—characterised by the action

$$S[\boldsymbol{\psi}, \boldsymbol{g}] = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}(\boldsymbol{\psi}, \boldsymbol{\nabla}\boldsymbol{\psi}, \boldsymbol{g}), \tag{1}$$

where g is the determinant of the metric, defined by

$$g = [\mu\nu\rho\sigma] g_{0\mu} g_{1\nu} g_{2\rho} g_{3\sigma} \tag{2}$$

with the permutation symbol $[\mu\nu\rho\sigma]$ such that $[0123] = 1$, while $\mathcal{L}$ denotes the covariant Lagrangian density of the matter field $\boldsymbol{\psi}$. The stress-energy tensor of $\boldsymbol{\psi}$ is the obtained by taking the variational derivative of the action S with respect to the metric:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad \text{that is} \qquad \delta S = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}. \tag{3}$$

In this exercise sheet, we propose to use this property to determine the stress-energy tensor of point particles, perfect fluids, and scalar fields.

1 From particles to fluids

In this exercise, we propose to determine the stress-energy tensor of a massive point particle, then a system of point particles, and see how we can recover the expression of the stress-energy tensor of a perfect fluid that you have seen in the course.

1.1 Point particle

Consider a point particle with mass m and neither spin not charge. The corresponding action is a function of the trajectory $\boldsymbol{y}(\lambda)$ of the particle through spacetime, as

$$S[\boldsymbol{y}, \boldsymbol{g}] = -m \int \mathrm{d}\tau = -m \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu} \dot{y}^\mu \dot{y}^\nu}, \tag{4}$$

where τ is the particle's proper time, while λ is an arbitrary parameter, and $\dot{y}^\mu \equiv \mathrm{d}y^\mu / \mathrm{d}\lambda$. If we want to extract the energy-stress tensor of such matter distribution, we have to face a first issue: the above integral concerns a single world-line, while eq. (1) is an integral over the whole spacetime, or at least over a 4-dimensional region of spacetime.

Q1. Show that one can actually write

$$S = \int \mathrm{d}^4x \, \mathcal{F}(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{y}}, \boldsymbol{g}), \tag{5}$$

where $\mathcal{F}$ is a distribution to be determined.

Q2. Take the variation of S with respect to a variation of the metric, and show that the resulting stress-energy tensor reads

$$T^{\mu\nu}(x^\mu) = \frac{p^\mu p^\nu}{p^0} \, \frac{\delta_\mathrm{D}[x^i - y^i(t)]}{\sqrt{-g}}, \tag{6}$$

where $t = x^0$, δ_D is the Dirac distribution, and $\boldsymbol{p}$ is the particle's four-momentum.

Q3. Interpret physically each term of the expression of T^{00}.

1.2 Fluid

Consider an ensemble of non-interacting point particles. Because they do not interact, the action of the system is just the sum of the actions of the individual particles, and hence so is its stress-energy tensor. Let $\mathcal{D}$ be a domain of $\mathcal{M}$ centred around an event E, and whose dimensions are small compared to the typical spacetime curvature radius. Within $\mathcal{D}$, all vectors can be considered to approximately belong to the same tangent space $\mathrm{T}_E\mathcal{M}$, in particular the four-momenta of the particles contained in $\mathcal{D}$, which can, thus, be summed.

Q4. Define the four-velocity $\bar{\boldsymbol{u}}$ of the barycentric frame of the particles in $\mathcal{D}$, as a function of their four-momenta.

Q5. Suppose that you work in a coordinate system adapted to this frame (e.g. the Fermi normal coordinate system associated with the associated world-line). Show that if $\boldsymbol{T}$ is smoothed over $\mathcal{D}$, then

$$T^{00} = n \, \langle \gamma m \rangle \tag{7}$$

$$T^{0i} = 0 \tag{8}$$

$$T^{ij} = n \langle \gamma m v^i v^j \rangle, \tag{9}$$

where $n = N_\mathcal{D}/V_\mathcal{D}$ is the number density of particles, and v^i the velocity of the particles, and γ the associated Lorentz factor.

Q6. In which limit does one recover the stress-energy tensor of a perfect fluid?

2 Variation of the metric determinant

The variation of a generic action

$$S = \int \mathrm{d}^4x \sqrt{-g} \, \mathcal{L} \tag{10}$$

for a variation $\delta\boldsymbol{g}$ of the metric reads

$$\delta S = \int \mathrm{d}^4x \; \delta\left(\sqrt{-g}\mathcal{L}\right) = \int \mathrm{d}^4x \; \left(\mathcal{L}\,\delta\sqrt{-g} + \sqrt{-g}\,\delta\mathcal{L}\right), \tag{11}$$

it is thus generally useful to determine the variation $\delta\sqrt{-g}$ of the metric determinant, which we propose to do in this exercise. For that purpose, let us consider $\boldsymbol{g}$ just like any invertible matrix.

Q1. Justify that, for a variation $\delta\boldsymbol{g}$ of $\boldsymbol{g}$, the variation of its determinant g can be written

$$\delta g = g \left[\det(\mathbf{1} + \boldsymbol{g}^{-1}\delta\boldsymbol{g}) - 1\right] \tag{12}$$

Q2. Show that, for any $n \times n$ matrix $\boldsymbol{M}$, $\det(\mathbf{1} + \varepsilon \boldsymbol{M}) = 1 + \varepsilon \, \mathrm{tr}\boldsymbol{M} + \mathcal{O}(\varepsilon^2)$.

Q3. Conclude that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{13}$$

3 Stress-energy tensor of a scalar field

The action of a scalar field ϕ minimally coupled to space-time geometry reads

$$S[\phi, \boldsymbol{g}] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right], \tag{14}$$

where V is the self-interaction potential of the field.

Q1. Show that its stress-energy tensor reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial^\rho\phi\partial_\rho\phi)g_{\mu\nu} - V(\phi)g_{\mu\nu}. \tag{15}$$

Q2. Justify that there always exists, locally, a coordinate system in which the stress-energy tensor of ϕ takes the same form as a perfect fluid. Identify the corresponding energy density and pressure. *Hint:* remember that any metric is locally flat.

Q3. Note that this pressure can be negative. Would such a thing be possible for a fluid made of massive particles? Do you know any context in which this property has been used?