

Geodesics and Parallel Transport

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1 Geodesics

Let γ be a curve on \mathcal{M} . Let us introduce a parameter λ such that $x^\mu(\lambda)$ represents the coordinates of \mathcal{M} . The tangent vector \mathbf{u} to γ is then such that $u^\mu = dx^\mu/d\lambda$. A *geodesic* is a self-parallel curve, i.e. whose tangent vector satisfies

$$\nabla_{\mathbf{u}}\mathbf{u} = f\mathbf{u}, \quad (1)$$

where f can be any function on \mathcal{M} .

Q1. Show that there exists a reparametrisation $\lambda \rightarrow s(\lambda)$ such that

$$\nabla_{\mathbf{v}}\mathbf{v} = \mathbf{0}, \quad (2)$$

where \mathbf{v} is the rescaled tangent vector, with $v^\mu \equiv dx^\mu/ds$.

Q2. A parameter s which satisfies the above property is called an *affine parameter*. Why?

Q3. Show that eq. (2) is equivalent to

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0. \quad (3)$$

In pseudo Riemannian geometry, a geodesic is also a *stationary curve*, i.e. a curve whose total length is stationary with respect to small modifications of its path. Consider, for simplicity, a space-like curve; its total length between points A and B is the functional

$$S[x^\mu] = \int_A^B ds = \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_A^B d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \quad (4)$$

Q4. Show that if S is stationary, that is if $\delta S/\delta x^\mu = 0$, then γ is indeed a self-parallel curve.

Q5. What is the geometric meaning of the affine parameter here?

Q6. How would the above change if we considered a time-like or a null curve instead?

2 Parallel transport

A vector field \mathbf{v} is said to be parallelly transported along a curve γ if its covariant derivative with respect to the tangent vector \mathbf{u} of γ is zero,

$$\nabla_{\mathbf{u}}\mathbf{v} = \mathbf{0}. \quad (5)$$

Q1. Show that a parallelly transported vector has its norm conserved along γ .

Q2. Show that if γ is an affinely parametrised geodesic, then the scalar product between \mathbf{u} and \mathbf{v} is conserved as well.

Consider the situation depicted on fig. 1. The vector \mathbf{v} , tangent to the sphere at its North pole, is parallelly transported along the sphere¹ along the closed curve γ , made of portions of great circles.

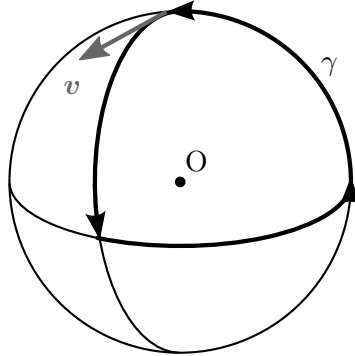


Figure 1 A vector \mathbf{v} is parallelly transported along the sphere following the closed curve γ .

Q3. Can you determine, without any calculation, what is going to be the state of \mathbf{v} after being parallelly transported along γ ? Well, if you are not sure... calculate!

3 Tensor: Some practise (more!)

This is a computational exercise. The goal is to practise the technical computations (Curvature, Parallel transport, etc.). You can skip it if you feel comfortable. **The three parts of the problem are independent.**

Part I

We consider the Cartesian coordinates $x^\mu = (t, x, y, z)$ and the metric

$$\mathbf{g} = -\mathbf{d}t^2 + f(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2), \quad f(t) > 0, \tag{6}$$

and the 4-vector

$$\mathbf{u} = \partial_t + v(t)\partial_x. \tag{7}$$

Q1. Compute the Christoffel symbols, the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$.

Hint: Compute the inverse metric $g^{\mu\nu}$ and use the symmetries of the problem to reduce the number of computations you need to do.

Q2. Compute $\nabla_{\mathbf{u}}\mathbf{u}$. Is this vector parallel transported?

Part II

We consider the unit sphere with coordinates $x^\mu = (\theta, \phi)$ and metric

$$\mathbf{d}s^2 = \mathbf{d}\theta^2 + \sin^2(\theta) + \mathbf{d}\phi^2 \tag{8}$$

We consider a particle whose trajectory is given by

$$x^\mu(\lambda) = (\theta(\lambda), \phi(\lambda)) = (\theta_0, \omega\lambda), \quad \theta_0 \in (0, \pi). \tag{9}$$

¹while keeping tangent to the sphere, i.e. the surface of the sphere is considered the manifold in which we define geometry and vector fields

Q3. Compute $\nabla_{\mathbf{u}}\mathbf{u}$, where

$$u^\mu = \frac{dx^\mu}{d\lambda}.$$

Q4. For which value(s) of θ_0 is \mathbf{u} parallel transported along itself? Is it surprising?

Part III

We consider a 4-dimensional Minkowski space, whose metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (10)$$

and a 3-dimensional hypersurface S implicitly given by

$$-t^2 + x^2 + y^2 + z^2 = -R^2. \quad (11)$$

A possible parametrisation is then

$$t = R \cosh \chi, \quad (12)$$

$$x = R \sinh \chi \sin \theta \cos \phi, \quad (13)$$

$$y = R \sinh \chi \sin \theta \sin \phi, \quad (14)$$

$$z = R \sinh \chi \cos \theta, \quad (15)$$

with $\chi \in \mathbb{R}_+$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.

Q5. Compute the induced metric on the hypersurface S , i.e. compute

$$ds^2 = g_{ab} dx^a dx^b, \quad (16)$$

with $x^a = (\chi, \theta, \phi)$.

Q6. Can you give an interpretation for the metric you obtained?